AN APPLICATION OF A POINTWISE VARIATIONAL PRINCIPLE IN ELASTODYNAMICS

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AN APPLICATION OF A POINTWISE VARIATIONAL PRINCIPLE IN ELASTODYNAMICS

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SUMMARY

A low frequency approximation for the response of the surface of an elastic half-space, due to the vertical oscillation of a rigid body with a flat circular base, is obtained by means of a variational principle which characterizes the displacement field at an arbitrary point on the free surface. A first variational approximation is obtained and checked for accuracy by means of a second variational approximation. This problem has many practical ramifications.

CHAPTER I

INTRODUCTION

We consider a mixed boundary value problem in elastodynamics, which is of interest in connection with seismology and the effect of vibrating machine foundations. This problem can be described physically in terms of a body with a flat circular base oscillating normal to the surface of an elastic half-space. We are interested in the displacement field on the surface of the half-space. Our model will consist of a rigid body with a flat frictionless base oscillating normal to the surface of a linear, homogeneous, isotropic half-space, which is otherwise free from tractions. The oscillations are assumed to have harmonic time dependence $e^{-i\omega t}$.

Our procedure is to construct a pointwise variational principle for the above displacement field, that is, a variational principle which has as its stationary values the components of displacement on the free surface of the half-space. This variational principle, which is similar to that previously constructed in [1], is based on an integral equation formulation of our problem and is discussed in Chapter II. Our results are valid in the "low" range of a certain frequency parameter and for <u>arbitrary</u> points on the free surface of the half-space, except in the immediate vicinity of the edge of the vibrating body.

The variational principle constructed in this thesis is only a stationary principle, and it is not possible to obtain error estimates using currently available procedures. To motivate the reasons for this difficulty we will compare certain properties associated with Laplace's equation and the Helmholtz equation in a half-space. This will enable us to anticipate the analogous differences to be expected when problems in a half-space associated with the differential equations governing elastostatics and harmonic elastodynamics are considered. For physical problems leading to Laplace's equation one is concerned with real-valued solutions. However, the Helmholtz equation, which arises in connection with harmonic wave phenomena, requires complex-valued solutions in a half-space. This is because of the necessity of imposing the Sommerfeld radiation condition which corresponds physically to the exclusion of waves coming from infinity. Hence an error estimate would necessarily involve error estimates for both the real and imaginary parts of the solution. The standard variational principle for Laplace's equation is Dirichlet's principle. This principle is a minimum principle and is associated with a positive definite metric. Hence there is a corresponding Schwarz inequality which is the basic tool used in obtaining error bounds. The analogous variational principle generating the Helmholtz equation is associated with an indefinite metric, and the corresponding stationary point does not correspond to either a maximum or a minimum of the associated functional. Furthermore, there is no Schwarz inequality, and therefore no bounds are possible. These difficulties are discussed by Dolph [2], [3], [4]. Bramble and Payne [5], based on a knowledge of the smallest eigenvalue of an operator associated with the governing differential equation, have, by means of a number of ingenious devices, developed a procedure for obtaining bounds in the Dirichlet problem for a class of elliptic second order partial differential equations with an associated metric of indefinite type. However, it should be pointed out that their method is not applicable to exterior problems, for example, half-space problems. A method for obtaining pointwise bounds on the value of the unknown solution of a boundary value problem, or on the derivatives of such a solution, governed by a linear elliptic equation with an associated quadratic form of positive definite type is presented in the work of Synge [6]. The method is motivated by geometry and employs generalized notions of circle, angle and vector projection. As Synge indicates in Part III of [6], for problems associated with an indefinite metric the concepts of generalized angle and vector projection are lost, and the notion of a circle is replaced by that of a hyperbola. Consequently, Synge's technique fails for problems associated with an indefinite metric.

We circumvented the corresponding difficulties in our elastodynamics problem by using two variational approximations. In the first such approximation the trial functions used were exact in the limit of zero frequency. To obtain our second variational approximation we took a set of trial functions containing parameters such that the resulting expression reduced to the first variational approximation when these parameters were set equal to zero. The results obtained from the second variational approximation were found to agree extremely well with the first for "low" frequencies, thereby providing a check on the first variational approximation. In addition the second variational approximation should lead to results valid over a much wider range of frequencies than the first.

The only work located that is related to our problem is contained in articles [7] and [8]. In [7] the authors obtain numerical results for the response of the free surface of an elastic half-space due to the oscillation of a rigid body of the type considered in this thesis, but in various modes of excitation. However, their results are based on a rather restrictive assumption, namely, that the distribution of stress under the oscillating body is the same as that in the corresponding static problem. Our procedure, while bearing some similarity to the method discussed in [7], is a basically different and more accurate approach to the problem under consideration. The authors of [8] also discuss the same problems as those discussed in [7]. Their procedure is based on integral equation formulations of the corresponding boundary value problem. However, only the far-field displacements are obtained.

CHAPTER II

CONSTRUCTION OF A POINTWISE VARIATIONAL PRINCIPLE

We shall construct a pointwise variational principle for the following problem in elastodynamics. Determine the displacement field within a half-space D, with boundary B, due to a rigid body with a flat base of arbitrary shape oscillating harmonically normal to its surface with zero shear stresses in the area B_u beneath the body and all stresses zero in $B_T = B - B_u$.

Let the half-space $D = \{x_1, x_2, x_3): x_3 > 0\}$[1] and take $n = (0,0,-1)$ as the unit normal vector to B. Let the components of displacement and stress at the point $P \in D + B$ be given by $u_i(P)$ and $\tau_{ij}(P)$, respectively.[2] Let the time dependent displacement of the body in the x_3-direction be $u_3(Q) = de^{-i\omega t}$, $Q \in B_u$.[3] Let λ and μ be the Lamé constants [9]. Then the physical problem described in the preceding paragraph corresponds to the boundary value problem

$$\mu u_{i,jj}(P) + (\lambda + \mu)u_{j,ji}(P) + \rho\omega^2 u_i(P) = 0, \quad P \in D \,, \tag{1a}$$

$$u_3(Q) = d \qquad , \quad Q \in B_u \,, \tag{1b}$$

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[1]Throughout this chapter we employ rectangular Cartesian coordinates.

[2]Unless otherwise indicated, here and in the sequel i, j and α each take on the values 1, 2 and 3.

[3]The factor $e^{-i\omega t}$ will be omitted in subsequent expressions.

$$\tau_{13}(Q) = \tau_{23}(Q) = 0 \quad , \qquad Q \in B_u \ , \qquad (1c)$$

$$\tau_{i3}(Q) = 0 \qquad\qquad , \qquad Q \in B_T \ , \qquad (1d)$$

together with an appropriate radiation condition to assure outgoing waves at infinity. Here repeated subscripts indicate summation over the range 1, 2, 3 and commas denote partial differentiation.

We now proceed to construct a functional which has as its stationary values the three components, $u_\alpha(P)$, $P \in D + B_T$, of the displacement field for the problem (1a,b,c,d). The resulting variational principle is a modification of that derived in [1].

In order to construct the required functional, we first develop three integral representations for $u_\alpha(P)$, $P \in D + B_T$.

To obtain our first integral representation, we let $\hat{u}^i_\alpha(P,Q)$ be the components of displacement in the x_α-direction $P \in D + B_T$ due to a unit oscillating force in the x_i-direction at $Q \in B_u$ with the boundary B otherwise free from tractions. In other words the associated traction $\hat{T}^i_\alpha(P,Q) = 0$, $Q \in B$, $P \in D + B - \{Q\}$. According to [1] we have the integral representation

$$u_\alpha(P) = - \int_{B_u} \hat{u}^3_\alpha(P,Q)\tau_{33}(Q)dA_Q \, , \ P \in D + B_T \quad , \qquad (2)$$

where $\tau_{33}(Q)$ is the stress normal to B_u at Q.

We now proceed with the construction of a second integral representation for $u_\alpha(P)$. Let $u^{*\alpha}_i(Q,P)$, $P \in D + B_T$, $Q \in B$, with associated tractions $T^{*\alpha}_i(Q,P)$, be the displacement in the x_i-direction at Q due to a unit oscillating point force in the x_α-direction at P with the specifications

$$u_3^{*\alpha}(Q,P) = 0, \; P \in B_T \; , \quad Q \in B_u \; , \tag{3}$$

$$T_1^{*\alpha}(Q,P) = T_2^{*\alpha}(Q,P) = 0, \; P \in D + B_T \; , \; Q \in B_u \; ,$$

and

$$T_i^{*\alpha}(Q,P) = 0, \; P \in D + B_T \; , \; Q \in B_T - \{Q\}$$

and an appropriate radiation condition at infinity. We note that the functions defined above are not the same as those defined in equation (82) of [1]. Proceeding by analogy with the development in [1]

$$u_\alpha(P) = \int_{B_u} T_1(Q)u_3^{*1}(Q,P)dA_Q + \int_{B_u} T_2(Q)u_3^{*2}(Q,P)dA_Q \tag{4}$$

$$- \int_{B_u} T_3^{*\alpha}(Q,P)u_3(Q)dA_Q + \int_{B_T} T_i(Q)u_i^{*\alpha}(Q,P)dA_Q$$

$$= - \int_{B_u} T_3^{*\alpha}(Q,P)u_3(Q)dA_Q \; , \; P \in D + B_T \; ,$$

since $T_i(Q) = \tau_{ij}(Q)n_j = 0$ for $Q \in B_T$ by (1d). Also, because $T_3^{*\alpha}(Q,P) = \tau_{3i}^{*\alpha}(Q,P)n_i = -\tau_{33}^{*\alpha}(Q,P)$, we have

$$u_\alpha(P) = \int_{B_u} \tau_{33}^{*\alpha}(Q,P) \, u_3(Q)dA_Q, \; P \in D + B_T \; . \tag{5}$$

For future reference we note that

$$\hat{u}_\alpha^3(P,Q) = \int_{B_u} \tau_{33}^{*\alpha}(R,P)\hat{u}_3^3(R,Q)dA_R, \; P \in D + B_T, \; Q \in B_u \; . \tag{6}$$

The derivation of this result is analogous to that for (5).

A third representation for the displacement field $u_\alpha(P)$ can be obtained by combining the representations (2) and (6) obtaining

$$u_\alpha(P) = - \int_{B_u}\int_{B_u} \overset{*\alpha}{\tau}_{33}(Q,P)\hat{u}^3_3(Q,R)\tau_{33}(R)dA_R\, dA_Q, \; P \in D + B_T \,. \qquad (7)$$

The integral representations (2), (5) and (7) motivate the construction of the stationary functional

$$\{u_\alpha(P)\} = \frac{\int_{B_u} \hat{u}^3_\alpha(P,Q)\overset{a}{\tau}_{33}(Q)dA_Q \cdot \int_{B_u} \overset{a*\alpha}{\tau}_{33}(Q,P)u_3(Q)dA_Q}{\int_{B_u}\int_{B_u} \overset{a*\alpha}{\tau}_{33}(Q,P)\hat{u}^3_3(Q,R)\; \overset{a}{\tau}_{33}(R)\; dA_R\; dA_Q} \;, \; P \in D + B_T \,, \qquad (8)$$

which gives the exact displacement field $u_\alpha(P)$, when the admissible (trial) functions $\overset{a}{\tau}_{33}(Q)$ and $\overset{a*\alpha}{\tau}_{33}(P,Q)$ are replaced by the exact dynamic stress distributions $\tau_{33}(Q)$ and $\overset{*\alpha}{\tau}_{33}(P,Q)$, respectively. The notation $\{u_\alpha(P)\}$ in (8) is used to indicate that the stationary value of the functional is $u_\alpha(P)$. We note that the functional (8) is homogeneous with respect to the admissible functions so that multiplicative constants are unessential.

We now proceed to show that the functional (8) is stationary with respect to variations in the vicinity of the exact dynamic stress distributions, $\tau_{33}(P,Q)$ and $\overset{*\alpha}{\tau}_{33}(P,Q)$, and that this stationary value is $u_\alpha(P)$. For this purpose we will consider the functional

$$\{u_\alpha(P)\} = -\int_{B_u} \hat{u}^3_\alpha(P,Q)\overset{a}{\tau}_{33}(Q)dA_Q + \int_{B_u} \overset{a}{\tau}{}^{*\alpha}_{33}(Q,P)u_3(Q)dA_Q \tag{9}$$

$$+\int_{B_u}\int_{B_u} \overset{a}{\tau}{}^{*\alpha}_{33}(Q,P)\hat{u}^3_3(Q,R)\overset{a}{\tau}_{33}(R)dA_R\, dA_Q,\ P \epsilon D + B_T \ ,$$

and observe that (8) is merely the homogeneous form of (9) and has the same stationary value.

Let

$$\overset{a}{\tau}_{33}(Q) - \tau_{33}(Q) = O(\epsilon) \tag{10}$$

and

$$\overset{a}{\tau}{}^{*\alpha}_{33}(Q,P) - \tau^{*\alpha}_{33}(Q,P) = O(\epsilon) \ . \tag{11}$$

We therefore require to show that

$$\{u_\alpha(P)\} - u_\alpha(P) = O(\epsilon^2),\ P \epsilon D + B_T \ . \tag{12}$$

We have

$$\{u_\alpha(P)\} - u_\alpha(P) \tag{13}$$

$$= [-\int_{B_u} \hat{u}^3_\alpha(P,Q)\overset{a}{\tau}_{33}(Q)dA_Q + \int_{B_u} \overset{a}{\tau}{}^{*\alpha}_{33}(Q,P)u_3(Q)dA_Q$$

$$+\int_{B_u}\int_{B_u} \overset{a}{\tau}{}^{*\alpha}_{33}(Q,P)\hat{u}^3_3(Q,R)\overset{a}{\tau}_{33}(R)dA_R\, dA_Q]$$

$$- [-\int_{B_u} \hat{u}_\alpha^3(P,Q)\tau_{33}(Q)dA_Q + \int_{B_u} \overset{*\alpha}{\tau}_{33}(Q,P)u_3(Q)\, dA_Q$$

$$+ \int_{B_u}\int_{B_u} \overset{*\alpha}{\tau}_{33}(Q,P)\hat{u}_3^3(Q,R)\tau_{33}(R)\, dA_R\, dA_Q]$$

$$= - \int_{B_u} \hat{u}_\alpha^3(P,Q)[\overset{a}{\tau}_{33}(Q) - \tau_{33}(Q)]\, dA_Q$$

$$+ \int_{B_u} [\overset{a*\alpha}{\tau}_{33}(Q,P) - \overset{*\alpha}{\tau}_{33}(Q,P)]u_3(Q)\, dA_Q$$

$$+ \int_{B_u}\int_{B_u} [\overset{a*\alpha}{\tau}_{33}(Q,P) - \overset{*\alpha}{\tau}_{33}(Q,P)]\hat{u}_3^3(Q,R)\tau_{33}(R)\, dA_R\, dA_Q$$

$$+ \int_{B_u}\int_{B_u} \overset{*\alpha}{\tau}_{33}(Q,P)\hat{u}_3^3(Q,R)[\overset{a}{\tau}_{33}(R) - \tau_{33}(R)]\, dA_R\, dA_Q$$

$$+ \int_{B_u}\int_{B_u} [\overset{a*\alpha}{\tau}_{33}(Q,P) - \overset{*\alpha}{\tau}_{33}(Q,P)]\, \hat{u}_3^3(Q,R)[\overset{a}{\tau}_{33}(R) - \tau_{33}(R)]\, dA_R\, dA_Q\, .$$

Using (6) we have

$$\int_{B_u} \hat{u}_\alpha^3(P,Q)[\overset{a}{\tau}_{33}(Q) - \tau_{33}(Q)]\, dA_Q \tag{14}$$

$$= \int_{B_u}\int_{B_u} \overset{*\alpha}{\tau}_{33}(R,P)\hat{u}_3^3(R,Q)[\overset{a}{\tau}_{33}(Q) - \tau_{33}(Q)]\, dA_R\, dA_Q$$

and (2) implies that

$$\int_{B_u} [\overset{a*\alpha}{\tau}_{33}(Q,P) - \overset{*\alpha}{\tau}_{33}(Q,P)]u_3(Q)\, dA_Q \tag{15}$$

$$= - \int_{B_u}\int_{B_u} [\overset{a*\alpha}{\tau}_{33}(Q,P) - \overset{*\alpha}{\tau}_{33}(Q,P)]\hat{u}_3^3(Q,R)\tau_{33}(R)\, dA_R\, dA_Q\, .$$

Combining equations (13), (14) and (15) we have

$$\{u_\alpha(P)\} - u_\alpha(P) \tag{16}$$

$$= \int_{B_u}\int_{B_u} [\overset{a*\alpha}{\tau}{}_{33}(Q,P) - \tau^{*\alpha}_{33}(Q,P)]\, \hat{u}^3_3(Q,R)[\overset{a}{\tau}_{33}(R) - \tau_{33}(R)]\, dA_R\, dA_Q$$

$$= O(\epsilon^2)$$

as was required.

We remark that the functional (8) has a saddle point when $\overset{a}{\tau}_{33}(Q) = \tau_{33}(Q)$ and $\overset{a*\alpha}{\tau}{}_{33}(P,Q) = \tau^{*\alpha}_{33}(P,Q)$, rather than a maximum or minimum, so that it is not possible to obtain bounds on approximations obtained by varying the trial functions about their stationary values.

For axisymmetric problems it is convenient to modify the fundamental singularities. For a discussion of this and an explanation of the notation, see Appendix A. With the resulting modifications, we obtain the homogeneous functionals

$$\{U_r(b)\} = \frac{\int_0^a \hat{U}^z_r(b,\rho)\overset{a}{\tau}_{zz}(\rho)\rho d\rho \cdot \int_0^a \overset{a*r}{\tau}{}_{zz}(\rho,b)U_z(\rho)\rho d\rho}{\int_0^a\int_0^a \overset{a*r}{\tau}{}_{zz}(\rho,b)\hat{U}^z_z(\rho,\rho')\overset{a}{\tau}_{zz}(\rho')\rho\rho' d\rho d\rho'} \tag{17}$$

and

$$\{U_z(b)\} = \frac{\int_0^a \hat{U}^z_z(b,\rho)\overset{a}{\tau}_{zz}(\rho)\rho d\rho \cdot \int_0^a \overset{a*z}{\tau}{}_{zz}(\rho,b)U_z(\rho)\rho d\rho}{\int_0^a\int_0^a \overset{a*z}{\tau}{}_{zz}(\rho,b)\hat{U}^z_z(\rho,\rho')\overset{a}{\tau}_{zz}(\rho)\rho\rho' d\rho d\rho'} \tag{18}$$

which are analogous to (8). Here the superscript a indicates an admissible (trial) function. The homogeneous functionals (17) and (18) will be used to obtain an approximate solution of the boundary value problem (1a,b,c,d) at points $P(b,\theta,0)$ in the special case where B_u is the circular region $0 \leq \rho \leq a$, $z = 0$. To this end we will choose the trial functions $\overset{a}{\tau}_{zz}(\rho)$, $\overset{a*z}{\tau}_{zz}(\rho,b)$ and $\overset{a*r}{\tau}_{zz}(\rho,b)$ so that they have certain known properties of the exact dynamic stress distributions; see Appendix C.

CHAPTER III

CONSTRUCTION OF TRIAL FUNCTIONS AND VARIATIONAL APPROXIMATIONS

For future reference we introduce the quantities h, k and γ according to

$$h = \frac{\omega}{c_1}\,, \quad k = \frac{\omega}{c_2} \quad \text{and} \quad \gamma = \frac{h}{k} < 1\,, \tag{19}$$

c_1 and c_2 being, respectively, the velocity of dilatation and equivoluminal waves and ω the circular frequency of oscillations, so that $k = 0$ yields the corresponding static problem.

For our first variational approximations we use as our admissible functions the exact <u>static</u> stress distributions corresponding to $\tau_{zz}(\rho)$, $\overset{*z}{\tau}_{zz}(\rho,b)$ and $\overset{*r}{\tau}_{zz}(\rho,b)$, which will subsequently be labeled $\overset{1}{\tau}_{zz}(\rho)$, $\overset{1*z}{\tau}_{zz}(\rho,b)$ and $\overset{1*r}{\tau}_{zz}(\rho,b)$, respectively. These stress distributions have singularities of the correct type at the edge of the disc; see Appendix C.

The trial function $\overset{1}{\tau}_{zz}(\rho)$ is given in [10] as

$$\overset{1}{\tau}_{zz}(\rho) = \frac{4\mu}{\pi}\,\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{1}{\sqrt{a^2-\rho^2}}\,, \quad 0 \le \rho < a\,.^{1} \tag{20}$$

[1]The sign is opposite to that given in [10] due to the choice of the half-space D.

The trial functions $\overset{1*z}{\tau}_{zz}(\rho,b)$ and $\overset{1*r}{\tau}_{zz}(\rho,b)$ can be obtained by solving a pair of Fredholm integral equations. These integral equations are obtained from the integral representations

$$\hat{U}_z^z(b,\rho')\big|_{k=0} = \int_0^a \overset{1*z}{\tau}_{zz}(\rho,b)\, \hat{U}_z^z(\rho,\rho')\big|_{k=0}\, \rho d\rho, b > a,\ \rho' < a, \quad (21)$$

and

$$\hat{U}_r^z(b,\rho')\big|_{k=0} = \int_0^a \overset{1*r}{\tau}_{zz}(\rho,b)\, \hat{U}_z^z(\rho,\rho')\big|_{k=0}\, \rho d\rho, b > a, \rho' < a, \quad (22)$$

which are adapted from Appendix A, equations (146) and (147), respectively.

From [9] the z-component of the displacement on B, at distance r from the origin, due to a force of unit strength in the z-direction at the origin is

$$\frac{\lambda + 2\mu}{4\pi\mu(\lambda + \mu)}\frac{1}{r} \quad .$$

Hence the z-component of the displacement on B at distance b from the origin due to a ring of unit point forces, acting in the z-direction at distance ρ' from the origin, is

$$\hat{U}_z^z(b,\rho')\big|_{k=0} = \frac{\lambda + 2\mu}{4\pi\mu(\lambda + \mu)} \int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + (\rho')^2 - 2b\rho'\cos(\theta - \varphi)}} \quad (23)$$

(See Figure 1). Our integral equation for $\overset{1*z}{\tau}_{zz}(\rho,b)$, obtained from (21) and (23), can now be written

$$\int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + \rho'^2 - 2b\rho'\cos(\theta - \varphi)}}$$

$$= \int_0^a \left[\int_0^{2\pi} \frac{d\varphi}{\sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\theta - \varphi)}}\right] \overset{1*z}{\tau}_{zz}(\rho,b)\rho d\rho, \quad \begin{array}{l} b > a, \\ \rho' < a, \end{array} \tag{24}$$

This integral equation for $\overset{1*z}{\tau}_{zz}(\rho,b)$ can be solved by a method due to Copson [11]. Using a result in [11], we can write

$$\int_0^a \left[\int_0^{2\pi} \frac{d\varphi}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\theta - \varphi)}}\right] \overset{1*z}{\tau}_{zz}(\rho,b)\rho d\rho \tag{25}$$

$$= \int_0^a \left[4 \int_0^{\min(\rho,\rho')} \frac{dt}{\sqrt{\rho^2 - t^2}\sqrt{\rho'^2 - t^2}}\right] \overset{1*z}{\tau}_{zz}(\rho,b)\rho d\rho$$

$$= 4 \int_0^{\rho'} \int_t^a \frac{\rho}{\sqrt{\rho^2 - t^2}\sqrt{(\rho')^2 - t^2}} \overset{1*z}{\tau}_{zz}(\rho,b)\rho d\rho dt$$

after a change of the order of integration. The integral equation (24) can now be written

$$\frac{1}{4}\int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + (\rho')^2 - 2b\rho'\cos(\theta - \varphi)}}$$

$$= \int_0^{\rho'} \frac{1}{\sqrt{(\rho')^2 - t^2}} \left[\int_t^a \frac{\overset{1*z}{\tau}_{zz}(\rho,b)\rho d\rho}{\sqrt{\rho^2 - t^2}}\right] dt, \quad b > a \;, \tag{26}$$

which is equivalent to the system of integral equations

$$S(t) = \int_t^a \frac{\rho}{\sqrt{\rho^2 - t^2}} \, {}^{1*z}_{\tau_{zz}}(\rho,b)\rho d\rho \tag{27}$$

and

$$\frac{1}{4}\int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + (\rho')^2 - 2b\rho'\cos(\theta - \varphi)}} = \int_0^{\rho'} \frac{S(t)\ dt}{\sqrt{(\rho')^2 - t^2}} . \tag{28}$$

We first solve equation (28) for $S(t)$. We can then substitute the result into equation (27) and solve for ${}^{1*z}_{\tau_{zz}}(\rho,b)$. Equation (28) is solved using [11] obtaining

$$S(\rho') = \frac{2}{\pi}\frac{d}{d\rho'}\int_0^{\rho'} \frac{t}{\sqrt{(\rho')^2 - t^2}} \left[\frac{1}{4}\int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + t^2 - 2bt\cos(\theta - \varphi)}}\right] dt . \tag{29}$$

This can be evaluated in closed form. From [12] we have

$$\int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + t^2 - 2bt\cos(\theta - \varphi)}} = \int_0^{2\pi}\int_0^{\infty} J_0\left(\tau\sqrt{b^2+t^2-2bt\cos(\theta - \varphi)}\right) d\tau d\varphi . \tag{30}$$

Using an addition formula for Bessel functions [12] we have

$$\int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + t^2 - 2bt\cos(\theta - \varphi)}} \tag{31}$$

$$= \int_0^{2\pi}\int_0^{\infty}\sum_{n=-\infty}^{\infty} J_n(b\tau)J_n(t\tau)\cos n(\theta - \varphi)d\tau d\varphi$$

$$= 2\pi\int_0^{\infty} J_0(b\tau)J_0(t\tau)d\tau$$

after term by term integration. Hence

$$S(\rho') = \frac{1}{2\pi}\frac{d}{d\rho'}\int_0^{\rho'} \frac{t}{\sqrt{(\rho')^2 - t^2}}\left[\int_0^{2\pi}\frac{d\varphi}{\sqrt{b^2 + t^2 - 2bt\cos(\theta - \varphi)}}\right]dt \qquad (32)$$

$$= \frac{d}{d\rho'}\int_0^{\rho'}\int_0^{\infty}\frac{t}{\sqrt{(\rho')^2 - t^2}} J_o(t\tau)\; J_o(b\tau)d\tau dt$$

$$= \frac{d}{d\rho'}\int_0^{\infty} J_o(b\tau)\left[\int_0^{\rho'}\frac{t\; J_o(t\tau)dt}{\sqrt{(\rho')^2 - t^2}}\right]d\tau$$

$$= \frac{d}{d\rho'}\int_0^{\infty}\frac{J_o(b\tau)\sin(\rho'\tau)d\tau}{\tau} = \int_0^{\infty} J_o(b\tau)\cos(\rho'\tau)d\tau$$

$$= \frac{1}{\sqrt{b^2 - (\rho')^2}} \quad ,$$

using results in [12] and [13]. We can now write (27) as

$$\frac{1}{\sqrt{b^2 - t^2}} = \int_t^a \frac{\overset{1*z}{\tau}_{zz}(\rho,b)\rho d\rho}{\sqrt{\rho^2 - t^2}} \quad . \qquad (33)$$

From [11] the solution of (33) is

$$\overset{1*z}{\tau}_{zz}(\rho,b) = -\frac{2}{\pi\rho}\frac{d}{d\rho}\int_\rho^a \frac{tdt}{\sqrt{t^2 - \rho^2}\sqrt{b^2 - t^2}} \qquad (34)$$

$$= \frac{2}{\pi}\sqrt{b^2 - a^2}\frac{\rho}{\sqrt{a^2 - \rho^2}}\frac{1}{(b^2 - \rho^2)}$$

Hence

$$\overset{1*z}{\tau}_{zz}(\rho,b) = \frac{1}{\sqrt{1-\left(\frac{\rho}{a}\right)^2}\left[\left(\frac{b}{a}\right)^2 - \left(\frac{\rho}{a}\right)^2\right]} \; , \; 0 \le \rho < a, \; b > a \; , \qquad (35)$$

except for a nonessential multiplicative constant.

The analogous integral equation for $\overset{*r}{\tau}_{zz}(\rho,b)$, $\rho < a$, $b > a$, is given in (22). To obtain $\hat{U}_r^z(b,\rho')|_{k=0}$ we use the expression [9]

$$-\frac{1}{4\pi(\lambda + \mu)}\frac{1}{r} \; , \qquad (36)$$

for the radial component of displacement on B, at a distance r from the origin, due to a force of unit strength in the z-direction at the origin. Referring to Figure 2, we see that CE = $b - \rho'\cos(\theta - \varphi)$ and therefore $\cos\psi = \frac{b - \rho'\cos(\theta - \varphi)}{R}$. Hence the radial component of displacement at $C(b,\theta,0)$ due to a unit force in the z-direction at $A(\rho',\varphi,0)$ is

$$-\frac{1}{4\pi(\lambda + \mu)} \; \frac{b - \rho'\cos(\theta - \varphi)}{b^2 + (\rho')^2 - 2b\rho'\cos(\theta - \varphi)} \; . \qquad (37)$$

Hence the radial component of the displacement on B at distance b from the origin due to a ring of unit forces in the z-direction on B acting at distance ρ' from the origin is

$$\hat{U}_r^z(b,\rho')\Big|_{k=0} = -\frac{1}{4\pi(\lambda + \mu)}\int_0^{2\pi} \frac{b - \rho'\cos(\theta - \varphi)}{b^2 + (\rho')^2 - 2b\rho'\cos(\theta - \varphi)}\, d\varphi \qquad (38)$$

$$= \begin{cases} 0\ , & b < \rho' \\ -\dfrac{1}{2(\lambda + \mu)b}\ , & b > \rho' \end{cases} .$$

Using the results (23) and (38) the integral equation (22) can be written

$$-\frac{1}{2(\lambda+\mu)b} = \int_0^a \left[\frac{\lambda \quad 2\mu}{4\pi\mu(\lambda+\mu)} \int_0^{2\pi} \frac{d\varphi}{\sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\theta - \varphi)}} \right] \tag{39}$$

$$\overset{1*r}{\tau}_{zz}(\rho,b)\rho d\rho,\ b > a\ .$$

Since

$$\int_0^a \left[\int_0^{2\pi} \frac{d\varphi}{\sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\theta - \varphi)}}\right] \overset{1*r}{\tau}_{zz}(\rho,b)\rho d\rho \tag{40}$$

$$= 4\int_0^a \left[\int_0^{\min(\rho',\rho)} \frac{dt}{\sqrt{(\rho^2 - t^2)((\rho')^2 - t^2)}}\right] \overset{1*r}{\tau}_{zz}(\rho,b)\rho d\rho$$

$$= 4\int_0^{\rho'} \int_t^a \frac{\rho' \tau^{1*r}_{zz}(\rho,b)d\rho dt}{\sqrt{(\rho^2 - t^2)((\rho')^2 - t^2)}}\ ,$$

using [11] and changing the order of integration, the integral equation (39) is equivalent to

$$-\frac{\pi}{2}\frac{\mu}{\lambda + 2\mu}\frac{1}{b} = \int_0^{\rho'}\int_t^a \frac{\rho\ \overset{1*r}{\tau}_{zz}(\rho,b)d\rho dt}{\sqrt{(\rho^2 - t^2)((\rho')^2 - t^2)}}\ ,\ b > a\ . \tag{41}$$

The integral equation (41) is equivalent to the system of integral equations

$$s(t) = \int_t^a \frac{\rho \, \overset{1*r}{\tau}_{zz}(b,\rho)d\rho}{\sqrt{\rho^2 - t^2}} \; , \; 0 \le t \le a, \; b > a, \tag{42}$$

and

$$-\frac{\pi}{2}\frac{1}{b}\frac{\mu}{\lambda + 2\mu} = \int_o^{\rho'} \frac{s(t)dt}{\sqrt{(\rho')^2 - t^2}} \; , \; 0 \le \rho' \le a. \tag{43}$$

Equation (43) has the solution [11]

$$s(\rho') = \frac{2}{\pi}\frac{d}{d\rho'}\int_o^{\rho'} \frac{t}{\sqrt{(\rho')^2 - t^2}}\left[-\frac{\pi}{2}\frac{1}{b}\frac{\mu}{\lambda + 2\mu}\right]dt = -\frac{1}{b}\frac{\mu}{\lambda + 2\mu} \, . \tag{44}$$

Using the result (44), equation (42) becomes

$$-\frac{1}{b}\frac{\mu}{\lambda + 2\mu} = \int_t^a \frac{\rho \, \overset{1*r}{\tau}_{zz}(\rho,b)d\rho}{\sqrt{\rho^2 - t^2}} \, . \tag{45}$$

From [11] we have

$$\overset{1*r}{\tau}_{zz}(\rho,b) = -\frac{2}{\pi}\frac{1}{b}\frac{\mu}{\lambda + 2\mu}\frac{1}{\sqrt{a^2 - \rho^2}} \; , \; 0 \le \rho < a, \; b > a \, . \tag{46}$$

The trial functions for the second variational approximation were constructed so that they possessed the correct singularities at the edge of the disc and in addition were functions of ρ^2; see

Appendix C. A set of functions with these properties is

$$\overset{2r}{\tau}_{zz}(\rho) = \overset{1}{\tau}_{zz}(\rho)(1 + c_1\rho^2) \tag{47}$$

$$\overset{2z}{\tau}_{zz}(\rho) = \overset{1}{\tau}_{zz}(\rho)(1 + c_2\rho^2) \tag{48}$$

$$\overset{2*r}{\tau}_{zz}(\rho,b) = \overset{1*r}{\tau}_{zz}(\rho,b)(1 + c_3\rho^2) \tag{49}$$

$$\overset{2*z}{\tau}_{zz}(\rho,b) = \overset{1*z}{\tau}_{zz}(\rho,b)(1 + c_4\rho^2) \tag{50}$$

where c_1, c_2, c_3 and c_4 are parameters which will eventually be chosen to be functions of the frequency parameter k; $\overset{2r}{\tau}_{zz}(\rho)$ and $\overset{2z}{\tau}_{zz}(\rho)$ are the trial functions corresponding to the admissible function $\overset{a}{\tau}_{zz}(\rho)$ in the second variational approximations for $U_r(b)$ and $U_z(b)$, respectively, and $\overset{2*r}{\tau}_{zz}(\rho,b)$ and $\overset{2*z}{\tau}_{zz}(\rho,b)$ are the trial functions corresponding to the functions $\overset{a*r}{\tau}_{zz}(\rho,b)$ and $\overset{a*z}{\tau}_{zz}(\rho,b)$, respectively. We note that the trial functions for the first variational approximation are special cases of those constructed above for the second variational approximation.

We now proceed to determine the points (c_1^s, c_3^s) and (c_2^s, c_4^s) which are stationary points for the functionals

$$\{U_r(b)\} = \frac{\int_0^a \hat{U}_r^z(\rho,b)\overset{2r}{\tau}_{zz}(\rho)\rho d\rho \cdot \int_0^a \overset{2*r}{\tau}_{zz}(\rho,b)U_z(\rho)\rho d\rho}{\int_0^a \int_0^a \overset{2*r}{\tau}_{zz}(\rho,b)\hat{U}_z^z(\rho,\rho')\overset{2r}{\tau}_{zz}(\rho')\rho\rho' d\rho d\rho'} \tag{51}$$

and

$$\{U_z(b)\} = \frac{\int_0^a \hat{U}_z^z(\rho,b)\overset{2z}{\tau}_{zz}(\rho)\rho d\rho \cdot \int_0^a \overset{2*z}{\tau}_{zz}(\rho,b)U_z(\rho)\rho d\rho}{\int_0^a \int_0^a \overset{2*z}{\tau}_{zz}(\rho,b)\hat{U}_z^z(\rho,\rho')\overset{2z}{\tau}_{zz}(\rho')\rho\rho' d\rho d\rho'} , \tag{52}$$

which are obtained from (17) and (18) respectively. Solving the system of algebraic equations

$$\frac{\partial}{\partial c_1} \{U_r(b)\}(c_1^s, c_3^s) = 0 \tag{53}$$

$$\frac{\partial}{\partial c_3} \{U_r(b)\}(c_1^s, c_3^s) = 0$$

$$\frac{\partial}{\partial c_2} \{U_z(b)\}(c_2^s, c_4^s) = 0$$

$$\frac{\partial}{\partial c_4} \{U_z(b)\}(c_2^s, c_4^s) = 0$$

we obtain

$$c_1^s = \frac{I_1 I_{13} - I_9 I_2}{I_{14} I_2 - I_1 I_{10}} \tag{54}$$

$$c_2^s = \frac{I_5 I_{14} - I_9 I_6}{I_{13} I_6 - I_5 I_{10}} \tag{55}$$

$$c_3^s = \frac{I_3 I_{15} - I_{11} I_4}{I_{12} I_4 - I_3 I_{16}} \tag{56}$$

and

$$c_4^s = \frac{I_7 I_{12} - I_{11} I_8}{I_{15} I_8 - I_7 I_{16}} \tag{57}$$

where

$$I_1 = \int_o^a \overset{1*r}{\tau}_{zz}(\rho, b) U_z(\rho) \rho d\rho \tag{58}$$

$$I_2 = \int_o^a \overset{1*r}{\tau}_{zz}(\rho, b) U_z(\rho) \rho^3 d\rho \tag{59}$$

$$I_3 = \int_o^a \overset{1*z}{\tau}_{zz}(\rho, b) U_z(\rho) \rho d\rho \tag{60}$$

$$I_4 = \int_o^a \overset{1*z}{\tau}_{zz}(\rho, b) U_z(\rho) \rho^3 d\rho \tag{61}$$

$$I_5 = \int_o^a \hat{U}_r^z(b, \rho) \overset{1}{\tau}_{zz}(\rho) \rho d\rho \tag{62}$$

$$I_6 = \int_o^a \hat{U}_r^z(b, \rho) \overset{1}{\tau}_{zz}(\rho) \rho^3 d\rho \tag{63}$$

$$I_7 = \int_o^a \hat{U}_z^z(b, \rho) \overset{1}{\tau}_{zz}(\rho) \rho d\rho \tag{64}$$

$$I_8 = \int_o^a \hat{U}_z^z(b, \rho) \overset{1}{\tau}_{zz}(\rho) \rho^3 d\rho \tag{65}$$

$$I_9 = \int_0^a \int_0^a \overset{1}{\tau}{}^{*r}_{zz}(\rho',b)\hat{U}^z_z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho\rho' d\rho d\rho' \quad (66)$$

$$I_{10} = \int_0^a \int_0^a \overset{1}{\tau}{}^{*r}_{zz}(\rho',b)\hat{U}^z_z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho^3(\rho')^3 d\rho d\rho' \quad (67)$$

$$I_{11} = \int_0^a \int_0^a \overset{1}{\tau}{}^{*z}_{zz}(\rho',b)\hat{U}^z_z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho\rho' d\rho d\rho' \quad (68)$$

$$I_{12} = \int_0^a \int_0^a \overset{1}{\tau}{}^{*z}_{zz}(\rho',b)\hat{U}^z_z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho'\rho^3 d\rho d\rho' \quad (69)$$

$$I_{13} = \int_0^a \int_0^a \overset{1}{\tau}{}^{*r}_{zz}(\rho',b)\hat{U}^z_z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho^3\rho' d\rho d\rho' \quad (70)$$

$$I_{14} = \int_0^a \int_0^a \overset{1}{\tau}{}^{*r}_{zz}(\rho',b)\hat{U}^z_z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho(\rho')^3 d\rho d\rho' \quad (71)$$

$$I_{15} = \int_0^a \int_0^a \overset{1}{\tau}{}^{*z}_{zz}(\rho',b)\hat{U}^z_z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho(\rho')^3 d\rho d\rho' \quad (72)$$

and

$$I_{16} = \int_0^a \int_0^a \overset{1}{\tau}{}^{*z}_{zz}(\rho',b)\hat{U}^z_z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho^3(\rho')^3 d\rho d\rho' \quad . \quad (73)$$

The first variational approximations for $U_r(b)$ and $U_z(b)$ are

$$\overset{1}{U}_r(b) = \frac{I_1 I_5}{I_9} \quad (74)$$

and

$$\overset{1}{U}_z(b) = \frac{I_3 I_7}{I_{11}} \ , \tag{75}$$

respectively, when the trial functions $\overset{1}{\tau}_{zz}(\rho)$, $\overset{1*r}{\tau}_{zz}(\rho',b)$ and $\overset{1*z}{\tau}_{zz}(\rho',b)$ are substituted into the homogeneous functionals (17) and (18).

The second variational approximations simplify considerably when the expressions (54), (55), (56), and (57) are substituted for c_1^s, c_2^s, c_3^s and c_4^s, respectively. The results are

$$\overset{2}{U}_r(b) = \frac{I_5 I_{14} I_2 - I_5 I_1 I_{10} + I_6 I_1 I_{13} - I_6 I_9 I_2}{I_{13} I_{14} - I_9 I_{10}} \tag{76}$$

and

$$\overset{2}{U}_z(b) = \frac{I_7 I_{12} I_4 + I_3 I_8 I_{15} - I_7 I_3 I_{16} - I_{11} I_4 I_8}{I_{15} I_{12} - I_{11} I_6} \tag{77}$$

CHAPTER IV

INTEGRALS

We now describe various techniques which were used to reduce the integrals I_1 to I_{16} to forms convenient for subsequent numerical evaluation. For purposes of checking, the nonessential multiplicative constants have been retained.

Four of these integrals may be evaluated in closed form and therefore do not require the use of numerical quadrature. They are

$$I_1 = -\frac{2\gamma^2 ad}{\pi b} , \tag{78}$$

$$I_2 = -\frac{4\gamma^2 a^3 d}{3\pi b} , \tag{79}$$

$$I_3 = \frac{2d}{\pi} \arcsin\left(\frac{a}{b}\right) , \tag{80}$$

and

$$I_4 = \frac{2da^2}{\pi}\left[\left(\frac{b}{a}\right)^2 \arcsin\left(\frac{a}{b}\right) - \sqrt{\left(\frac{b}{a}\right)^2 - 1}\right] \tag{81}$$

where γ is defined in (19).

It is of interest to note that the above expression (78) for I_1 is exactly the radial displacement $U_r(b)$ on the plane B at $r = b$ for the problem referred to at the end of Chapter II in the static case. This is indeed to be expected if one compares the integral representation

(143) with the definition of I_1 as given by (58). By comparing (132) and (60) a similar statement can be made for I_3 given in (80). These observations provide a check on the trial functions $\overset{1*z}{\tau}_{zz}(\rho,b)$ and $\overset{1*r}{\tau}_{zz}(\rho,b)$ which are given by (34) and (46), respectively.

We were not able to evaluate the integrals I_5 to I_{16} in closed form. Because the fundamental singularities $\hat{U}^z_z(b,\rho)$ and $\hat{U}^z_r(b,\rho)$ as given by (160) and (165), respectively, are expressed in terms of integrals from zero to infinity and, since the integrands have a variety of singularities, special techniques have been employed to put the integrals in forms convenient for numerical computation.

We let

$$t = \frac{b}{a} > 1 \ , \tag{82}$$

and

$$\hat{x} = kax \ , \tag{83}$$

where k is defined in (19). We also let

$$f_1(s) = 2s(2s^2-1) - 2s\sqrt{s^2-1}\ \sqrt{s^2-\gamma^2} - \frac{s^3\sqrt{s^2-1}}{\sqrt{s^2-\gamma^2}} - \frac{s^3\sqrt{s^2-\gamma^2}}{\sqrt{s^2-1}} \tag{84}$$

where s > 1 is the location of a Rayleigh pole [14],

$$f_2(x) = \left(x^2 + \frac{1}{2}\right)^2 - x^2\sqrt{x^2+\gamma^2}\ \sqrt{x^2+1} \ , \tag{85}$$

$$\hat{s} = kas \ , \tag{86}$$

and

$$K(x) = \begin{cases} \dfrac{x\sqrt{\gamma^2-x^2}}{\left(x^2-\frac{1}{2}\right)^2 + x^2\sqrt{\gamma^2-x^2}\,\sqrt{1-x^2}} \,, & 0 \le x \le \gamma \\[2ex] \dfrac{x^3(x^2-\gamma^2)\sqrt{1-x^2}}{\left(x^2-\frac{1}{2}\right)^4 + x^4(x^2-\gamma^2)(1-x^2)} \,, & \gamma < x \le 1 \,, \end{cases} \tag{87}$$

with γ as in (19). To avoid possible confusion we note that $K_0(z)$ is a modified Bessel function. Furthermore, we let

$$q(\hat{x}) = \frac{1 + \hat{x}^2 + \frac{2}{3}i\hat{x}^3 + \frac{4}{15}i\hat{x}^5 + (\hat{x}^2 + 2i\hat{x} - 1)e^{2i\hat{x}}}{\hat{x}^6} \,, \tag{88}$$

$$Q(\hat{x}) = -\frac{i}{2}\left[\frac{1-e^{2i\hat{x}}}{\hat{x}}\right] + 2e^{i\hat{x}}\left[\frac{\sin\hat{x}}{\hat{x}^3} - \frac{\cos\hat{x}}{\hat{x}^2}\right] - \frac{i}{2}\,\hat{x}q(\hat{x}) \,, \tag{89}$$

$$\hat{K}(x) = \frac{x\left(x^2-\frac{1}{2}\right)\sqrt{1-x^2}\,\sqrt{x^2-\gamma^2}}{\left(x^2-\frac{1}{2}\right)^4 + x^4(x^2-\gamma^2)(1-x^2)} \,, \quad \gamma \le x \le 1 \,, \tag{90}$$

$$T(\hat{x}) = \frac{\hat{x}\cos\hat{x} + (\hat{x}^2-1)\sin\hat{x}}{\hat{x}^2} \,, \tag{91}$$

$$T_1(\hat{x}) = \frac{1-e^{2i\hat{x}}}{\hat{x}} \,, \tag{92}$$

$$V(\hat{x},u) = i\pi[1 - e^{i\hat{x}}J_0(u\hat{x})] \,, \tag{93}$$

$$S(\hat{x}) = 2e^{i\hat{x}}\left(\frac{2}{i\hat{x}} + \frac{2}{\hat{x}^2} + \frac{2i}{\hat{x}^3}\right) \quad , \tag{94}$$

$$S_1(\hat{x}) = \frac{\hat{x}\cosh\hat{x} + (\hat{x}+1)\sinh\hat{x}}{\hat{x}^2} \quad , \tag{95}$$

and

$$T_2(u,\hat{x}) = -\frac{2}{i\hat{x}} - \frac{4i}{\hat{x}^3} - \frac{u^2}{i\hat{x}} + S(\hat{x})J_o(u\hat{x}) \quad . \tag{96}$$

We also note that λ, μ and γ obey the identities [9]

$$\frac{\lambda + \mu}{\lambda + 2\mu} = 1 - \gamma^2 \tag{97}$$

and

$$\frac{\mu}{\lambda + 2\mu} = \gamma^2 \quad . \tag{98}$$

The integrals I_5 to I_{13} follow, grouped according to the method used to put them in forms suitable for subsequent numerical quadrature. An example of each method is provided in Appendix B.

Method 1:

$$I_5 = \frac{d}{\pi}(1 - \gamma^2)\Big[-i\int_\gamma^1 \hat{K}(x)\, H_1^{(1)}(t\hat{x})\sin\hat{x}\, dx \tag{99}$$

$$+ \frac{\pi i s(2s^2 - 1 - 2\sqrt{s^2 - \gamma^2}\sqrt{s^2 - 1})H_1^{(1)}(t\hat{s})\sin\hat{s}}{f_1(s)}\Big]$$

$$I_6 = \frac{d}{\pi k^2}(1 - \gamma^2)\Big[-i\int_\gamma^1 \frac{\hat{K}(x)H_1^{(1)}(t\hat{x})T(\hat{x})dx}{x^2} \tag{100}$$

$$+ \frac{\pi i\left(2s^2 - 1 - 2\sqrt{s^2 - \gamma^2}\sqrt{s^2 - 1}\right)H_1^{(1)}(t\hat{s})T(\hat{s})}{f_1(s)}\Big]$$

Method 2:

$$I_7 = -\frac{d}{\pi}(1 - \gamma^2)\ \Big\{-\frac{2}{\pi}\int_0^\infty \frac{\sqrt{x^2 + \gamma^2}\ K_o(t\hat{x})\sinh\hat{x}}{f_2(x)}dx \tag{101}$$

$$-\ i\int_0^1 \frac{K(x)H_o^{(1)}(t\hat{x})\sin\hat{x}}{x}dx + \frac{\pi i\sqrt{s^2 - \gamma^2}\ H_o^{(1)}(t\hat{s})\sin\hat{s}}{f_1(s)}\Big\}$$

$$I_8 = -\frac{da^2}{\pi}(1 - \gamma^2)\ \Big\{-\frac{2}{\pi}\int_0^\infty \frac{\sqrt{x^2+\gamma^2}\ K_o(t\hat{x})}{f_2(x)}S_1(\hat{x})dx \tag{102}$$

$$-\ i\int_0^1 \frac{K(x)H_o^{(1)}(tx)T(x)}{x}dx + \frac{\pi i\sqrt{s^2 - \gamma^2}\ H_o^{(1)}(t\hat{s})T(\hat{s})}{f_1(s)}\Big\}$$

Method 3:

$$I_9 = -\frac{d}{\pi^2 t}\gamma^2(1 - \gamma^2)\Big[i\int_0^1 \frac{K(x)T_1(\hat{x})}{x}dx - \frac{\pi i\sqrt{s^2 - \gamma^2}\ T_1(\hat{s})}{f_1(s)}\Big] \tag{103}$$

Method 4:

$$I_{10} = \frac{2}{\pi^2}\frac{da^5}{b}\gamma^2(1-\gamma^2)\left[\int_o^1 \frac{K(x)Q(\hat{x})}{x}\,dx - \frac{\pi\sqrt{s^2-\gamma^2}\,Q(\hat{s})}{f_1(s)} - \frac{2\pi}{15}\frac{1}{1-\gamma^2}\right] \tag{104}$$

Method 5:

$$I_{11} = -\frac{2d}{\pi^2}(1-\gamma^2)\sqrt{t^2-1}\int_o^1 \frac{u}{\sqrt{1-u^2}\,[t^2-u^2]} \left\{\frac{i}{\pi}\left[\int_o^1 \frac{K(x)V(\hat{x},u)}{x}\,dx - \pi\frac{\sqrt{s^2-\gamma^2}\,V(\hat{s},u)}{f_1(s)}\right] - \frac{\pi}{1-\gamma^2}\right\} du \tag{105}$$

$$I_{12} = -\frac{da^2}{\pi}\sqrt{t^2-1}\int_o^1 \frac{u}{\sqrt{1-u^2}\,[t^2-u^2]} \left\{(1-\gamma^2)\,i\,\frac{ak}{2\pi}\left[-\int_o^\gamma K(x)T_2(u,\hat{x})dx - \int_\gamma^1 K(x)T_2(u,\hat{x})dx + \frac{\pi s\sqrt{s^2-\gamma^2}}{f_1(s)}T_2(u,\hat{s})\right] - \left(1+\frac{1}{2}u^2\right)\right\} du \tag{106}$$

$$I_{13} = \frac{da^2}{t}\gamma^2\int_o^1 \frac{u}{\sqrt{1-u^2}}\left\{(1-\gamma^2)\ i\ \frac{ak}{2\pi}\left[-\int_o^\gamma K(x)T_2(u,\hat{x})dx\right.\right. \tag{107}$$

$$- \int_{\gamma}^{1} K(x)T_2(u,\hat{x})dx + \frac{\pi s\sqrt{s^2 - \gamma^2}}{f_1(s)} T_2(u,\hat{s}) \Big] - \left(1 + \frac{1}{2} u^2\right)\Big\}du$$

The remaining three integrals, I_{14}, I_{15} and I_{16}, can be expressed in terms of integrals which have previously been considered, as follows:

$$I_{14} = I_{13} \tag{108}$$

$$I_{15} = b^2 I_{11} + a^2 b \sqrt{b^2 - a^2} \frac{I_9}{\gamma^2} \tag{109}$$

$$I_{16} = b^2 I_{12} + \frac{a^2 b\sqrt{b^2 - a^2}}{\gamma^2} I_{13}. \tag{110}$$

CHAPTER V

NUMERICAL INTEGRATION TECHNIQUES

A number of special numerical quadrature techniques were used in approximating the integrals I_5 to I_{13}.

In those integrals whose integrands involved $K(x)$, the derivative of the integrand became unbounded in the vicinity of γ and 1. The trapezoid rule was discovered to be more satisfactory computationally in the neighborhood of these points. Hence, the trapezoid rule was applied on the intervals $[\gamma - .05, \gamma + .05]$ and $[.95, 1]$. Simpson's Rule was used elsewhere.

The integrals I_7 and I_8 were reduced to sums of integrals, in each case including an infinite integral with an exponentially decaying integrand. For example, the integral I_7 involved the infinite integral

$$\int_0^\infty \frac{K_o(t\hat{x})\sinh \hat{x}\,\sqrt{x^2 + \gamma^2}\,dx}{\left(x^2 + \frac{1}{2}\right)^2 - x^2(x^2 + \gamma^2)^{1/2}(x^2 + 1)^{1/2}} \quad . \tag{111}$$

If (111) is approximated by

$$\int_0^{\frac{50}{akt}} \frac{K_o(t\hat{x})(\sinh \hat{x})\sqrt{x^2 + \gamma^2}\,dx}{\left(x^2 + \frac{1}{2}\right)^2 - x^2\sqrt{x^2 + \gamma^2}\,\sqrt{x^2 + 1}} \quad , \tag{112}$$

the error is

$$E = \int_{\frac{50}{akt}}^{\infty} \frac{K_o(t\hat{x}) \sinh \hat{x} \sqrt{x^2 + \gamma^2}}{\left(x^2 + \frac{1}{2}\right)^2 - x^2\sqrt{x^2 + \gamma^2}\,\sqrt{x^2 + 1}}\, dx\ . \qquad (113)$$

Since

$$\frac{\sqrt{x^2 + \gamma^2}}{\left(x^2 + \frac{1}{2}\right)^2 - x^2\sqrt{x^2 + \gamma^2}\,\sqrt{x^2 + 1}} \sim \frac{2}{(1 - \gamma^2)x} \text{ as } x \to \infty \qquad (114)$$

and (see [15])

$$K_o(u) \sim \left(\frac{\pi}{2u}\right)^{1/2} e^{-u} \quad \text{as} \quad u \to \infty\ , \qquad (115)$$

$$E = \int_{\frac{50}{akt}}^{\infty} \frac{K_o(t\hat{x}) \sinh \hat{x} \sqrt{x^2 + \gamma^2}}{\left(x^2 + \frac{1}{2}\right)^2 - x^2\sqrt{x^2 + \gamma^2}\,\sqrt{x^2 + 1}}\, dx \qquad (116)$$

$$\approx \int_{\frac{50}{akt}}^{\infty} \left(\frac{\pi}{2t\hat{x}}\right)^{1/2} e^{-t\hat{x}} \left[\frac{e^{\hat{x}} - e^{-\hat{x}}}{2}\right] \frac{2}{(1 - \gamma^2)x}\, dx$$

$$= \left(\frac{\pi}{2kat}\right)^{1/2} \frac{1}{1 - \gamma^2} \int_{\frac{50}{akt}}^{\infty} \frac{1}{x^2} \left[e^{(1-t)\hat{x}} - e^{-(1+t)\hat{x}}\right] dx$$

$$\le \left(\frac{\pi}{2kat}\right)^{1/2} \frac{1}{1 - \gamma^2} \int_{\frac{50}{akt}}^{\infty} \left[e^{(1-t)\hat{x}} - e^{-(1+t)\hat{x}}\right] dx$$

if akt ≤ 50. The final expression in (116) is equal to

$$\left(\frac{\pi ak}{2t}\right)^{1/2} \frac{1}{1-\gamma^2} \{\exp[-50(1-1/t)] - \exp[-50(1+1/t]\} \tag{117}$$

which can clearly be neglected if t > 1.1. If akt > 50, the integral

$$\int_1^\infty \frac{K_o(t\hat{x}) \sinh \hat{x} \sqrt{x^2+\gamma^2}\, dx}{\left(x^2+\frac{1}{2}\right)^2 - x^2\sqrt{x^2+\gamma^2}\sqrt{x^2+1}} \tag{118}$$

can be neglected by a similar argument.

Although the integral (112) has a removable singularity at 0, the logarithmic behavior of $K_o(t\hat{x})$ as $x \to 0$ leads to computational difficulties due to roundoff error. We therefore use the following special numerical technique. If f(x) is continuous on $[0,\epsilon]$ and $c > \epsilon$ is constant, a form of the Mean Value Theorem for integrals yields

$$\int_0^\epsilon f(x)\log\left(\frac{x}{c}\right)dx = f(\xi) \int_0^\epsilon \log\left(\frac{x}{c}\right)dx = f(\xi)\ \epsilon \log\left(\frac{\epsilon}{ce}\right) \tag{119}$$

where ξ is a number in $(0,\epsilon)$. If $|f(x)| \leq .02$, $0 < x \leq .02$, and $\epsilon = .02$, the approximation

$$f(\xi)\epsilon \log\left(\frac{\epsilon}{ce}\right) \approx \epsilon\ f\left(\frac{\epsilon}{e}\right) \log\left(\frac{\epsilon}{ce}\right) \tag{120}$$

is sufficiently accurate. Hence, if $F(x) = f(x) \log\left(\frac{x}{c}\right)$, $x > 0$ we use the approximation

$$\int_0^{\epsilon} F(x)dx \approx \epsilon\, F\left(\frac{\epsilon}{e}\right) . \tag{121}$$

Similarly, computational roundoff error leads to difficulties with such expressions as

$$q(x) = \frac{1 + x^2 + \frac{2}{3}\, i\, x^3 + \frac{4}{15}\, i\, x^5 + (x^2 + 2i\, x - 1)e^{2ix}}{x^6}\ ^1 \tag{122}$$

as $x \to 0$. The remedy for this is to approximate the function by a Taylor polynomial when x is near 0, i.e.

$$q(x) \approx .8888888 - .4571424\, ix + .1777604\, x^2 + .0564372\, i\, x^3 . \tag{123}$$

The integrals I_{11}, I_{12} and I_{13} are of the form

$$\int_0^1 \frac{uf(u)du}{\sqrt{1 - u^2}} \quad , \quad t > 1 \quad , \tag{124}$$

where f(u) is continuous and bounded on (0,1]. The singularity at 1 was eliminated by making the transformation

$$u = \sin\theta .$$

The resulting integral was approximated using Simpson's Rule.

The above numerical techniques were used to construct the programs UR and UZ20 in Appendix D for the approximation of $U_r(b)$ and $U_z(b)$, respectively.

[1]This function occurs in the treatment of the integral I_{10}.

APPENDIX A

REDEFINITION OF THE FUNDAMENTAL SINGULARITIES FOR OUR AXISYMMETRIC PROBLEM

If B_u is a disc of radius $a > o$ centered at the origin in the plane $B = \{(x,y,z): z = 0\}$, $P = (r,\theta,0)$ and $Q = (\rho,\varphi,0)$ in cylindrical polar coordinates, the representation (2) becomes

$$u_\alpha(r,\theta,0) = -\int_o^{2\pi}\int_o^a \hat{u}_\alpha^3(r,\theta,0;\rho,\varphi,0)\tau_{33}(\rho,\varphi,0)\rho d\rho d\varphi \ . \tag{125}$$

We note that $\tau_{33}(\rho,\varphi,0)$ is independent of φ. For this reason we adopt the notation

$$\tau_{zz}(\rho) \equiv \tau_{33}(\rho,\varphi,0) \tag{126}$$

We shall now use (125) to reformulate definitions for the fundamental singularities.

It will be observed that the component of displacement in the azimuthal direction is zero because of the axisymmetry.

We will now direct our attention to the points $P_o = (b,\theta,0)$.

In view of the axisymmetry $u_3(b,\theta,0)$ is independent of θ, and hence we may adopt the notation

$$U_z(b) \equiv u_3(b,\theta,0) \ . \tag{127}$$

We let

$$\hat{U}_z^z(b,\rho) \equiv \int_o^{2\pi} \hat{u}_3^3(b,\theta,0;\rho,\varphi,0)d\varphi \tag{128}$$

which is independent of θ because $\hat{u}_3^3(b,\theta,0;\rho,\varphi,0)$ depends on θ and φ only in the combination $\theta - \varphi$. The definition (128) can be taken as a redefinition of a fundamental singularity for our axisymmetric problem and can be interpreted as the displacement in the z-direction of a point on the ring $r = b$, $z = 0$, due to a ring of unit point forces oscillating in the z-direction around the circle $r = \rho$, $z = 0$. Using (126), (127) and (128), we obtain from (125)

$$U_z(b) = - \int_0^a \hat{U}_z^z(b,\rho)\tau_{zz}(\rho)\rho d\rho \ . \tag{129}$$

Taking $\alpha = 3$, $P = P_o$ and $Q = (\rho,\varphi,0)$, equation (4) can be written

$$u_3(b,\theta,0) = - \int_0^{2\pi} \int_0^a T_3^{*3}(\rho,\varphi,0;b,\theta,0)u_3(\rho,\varphi,0)\rho d\rho d\varphi \ . \tag{130}$$

We define $T_z^{*z}(\rho,b)$ by

$$T_z^{*z}(\rho,b) \equiv \int_0^{2\pi} T_3^{*3}(\rho,\varphi,0;b,\theta,0)d\varphi, \ b > a, \ 0 \le \rho < a \ , \tag{131}$$

using the observation that $T_3^{*3}(\rho,\varphi,0;b,\theta,0)$ depends on θ and φ only in the combination $\theta - \varphi$. Using definitions (127) and (131), we obtain from (130)

$$U_z(b) = - \int_0^a T_z^{*z}(\rho,b)U_z(\rho)\rho \, d\rho \ . \tag{132}$$

Employing an argument analogous to the one given for (125), we have

$$u_r(b,\theta,0) = -\int_0^{2\pi}\int_0^a \hat{u}_r^3(b,\theta,0;\rho,\varphi,0)\tau_{33}(\rho,\varphi,0)\rho d\rho d\varphi \ , \ b > a \tag{133}$$

where $u_r(b,\theta,0)$ is the radial component of displacement at $(b,\theta,0)$ due to the stress distribution $\tau_{33}(\rho,\varphi,0) \equiv \tau_{zz}(\rho)$ and $\hat{u}_r^3(b,\theta,0;\rho,\varphi,0)$ is the radial component of displacement at $(b,\theta,0)$ due to a ring of unit point forces oscillating in the z-direction around the circle $r = \rho$, $z = 0$. The symmetry implies that $u_r(b,\theta,0)$ is independent of θ, and hence we adopt the notation

$$U_r(b) \equiv u_r(b,\theta,0) \quad . \tag{134}$$

We define the fundamental singularity $\hat{U}_r^z(b,\rho)$ by

$$\hat{U}_r^z(b,\rho) \equiv \int_0^{2\pi} \hat{u}_r^3(b,\theta,0;\rho,\varphi,0)d\varphi \tag{135}$$

with an interpretation analogous to the one given for (128). Using (126), (134) and (136), (133) gives

$$U_r(b) = -\int_0^a \hat{U}_r^z(b,\rho)\tau_{zz}(\rho)\rho d\rho \ . \tag{136}$$

We now proceed to construct another integral representation for $U_r(b)$. Let $u_i^{*r}(Q,P)$, $P \in B_T$, $Q \in B$, with associated tractions $T_i^{*r}(Q,P)$, be the component of displacement in the x_i-direction at Q due to a unit oscillating point force in the radial direction at P with the specifications

$$u_3^{*r}(Q,P) = 0, \ P \in B_T, \ Q \in B_u \quad , \tag{137}$$

$$T_1^{*r}(Q,P) = T_2^{*r}(Q,P) = 0 \ , \quad P \in B_T \ , \quad Q \in B_u \ , \tag{138}$$

and

$$T_i^{*r}(Q,P) = 0 \ , \quad P \in B_T \ , \quad Q \in B_T - \{P\} \ . \tag{139}$$

Then an argument analogous to that given for (4) leads to

$$u_r(P) = - \int_{B_u} T_3^{*r}(Q,P) u_3(Q) dA_Q \ . \tag{140}$$

Taking $P = (b,\theta,0)$, $Q = (\rho,\varphi,0)$ and B_u the disc $\rho \le a$, (140) becomes

$$u_r(b,\theta,0) = - \int_0^{2\pi} \int_0^a T_3^{*r}(\rho,\varphi,0;b,\theta,0) u_3(\rho,\varphi,0) \rho d\rho d\varphi \ . \tag{141}$$

Since $T_3^{*r}(\rho,\varphi,0;b,\theta,0)$ depends on φ and θ only in the combination $\theta - \varphi$, we can write

$$T_z^{*r}(\rho,b) := \int_0^{2\pi} T_3^{*r}(\rho,\varphi,0;b,\theta,0) d\varphi \ . \tag{142}$$

Using (127), (134) and (142), (141) gives

$$U_r(b) = - \int_0^a T_z^{*r}(\rho,b) U_z(\rho) \rho d\rho \ , \tag{143}$$

which is a representation analogous to (4).

The representations (129) and (132) can be combined to obtain the representation

$$U_z(b) = \int_0^a \int_0^a T_z^{*z}(\rho,b) \ \vartheta_z^z(\rho,\rho') \tau_{zz}(\rho') \rho\rho' d\rho d\rho' \tag{144}$$

and, in a similar manner, we obtain

$$U_r(b) = \int_0^a \int_0^a T_z^{*r}(\rho,b)\hat{U}_z^z(\rho,\rho')\tau_{zz}(\rho')\rho\rho' d\rho d\rho' \tag{145}$$

from (143) and (129).

We note that

$$T_z^{*z}(\rho,b) = - \tau_{zz}^{*z}(\rho,b)$$

and

$$T_z^{*r}(\rho,b) = - \tau_{zz}^{*r}(\rho,b)$$

and, hence, that the representations (132), (143), (144) and (145) can be written

$$U_z(b) = \int_0^a \tau_{zz}^{*z}(\rho,b)U_z(\rho)\rho d\rho \ , \tag{146}$$

$$U_r(b) = \int_0^a \tau_{zz}^{*r}(\rho,b)U_z(\rho)\rho d\rho \ , \tag{147}$$

$$U_z(b) = - \int_0^a \int_0^a \tau_{zz}^{*z}(\rho,b)\hat{U}_z^z(\rho,\rho')\tau_{zz}(\rho')\rho\rho' d\rho d\rho' \tag{148}$$

and

$$U_r(b) = - \int_0^a \int_0^a \tau_{zz}^{*r}(\rho,b)\hat{U}_z^z(\rho,\rho')\tau_{zz}(\rho')\rho\rho' d\rho d\rho' \tag{149}$$

respectively.

We now proceed to obtain the resulting expressions for $\hat{U}_z^z(b,\rho)$ and $\hat{U}_r^z(b,\rho)$. From the classical result of Lamb [16], which is quoted in [17], we have

$$\hat{u}_3^3(r,\theta,0) = -\frac{k}{8\pi\mu}\int_0^\infty \frac{x\alpha(x)J_0(krx)}{f(x)}\,dx \tag{150}$$

and

$$\hat{u}_1^3(r,\theta,0) = \frac{k}{8\pi\mu}\int_0^\infty \frac{x^2[2x^2-1-2\alpha(x)\beta(x)]J_1(krx)}{f(x)}\,dx \tag{151}$$

where

$$\alpha(x) = \begin{cases} \sqrt{x^2-\gamma^2}, & x > \gamma \\ -i\sqrt{\gamma^2-x^2}, & -\gamma \le x \le \gamma \\ -\sqrt{x^2-\gamma^2}, & x < -\gamma, \end{cases} \tag{152}$$

$$\beta(x) = \begin{cases} \sqrt{x^2-1}, & x > 1 \\ -i\sqrt{1-x^2}, & -1 \le x \le 1 \\ -\sqrt{x^2-1}, & x < -1 \end{cases} \tag{153}$$

and

$$f(x) = \left(x^2 - \frac{1}{2}\right)^2 - x^2\,\alpha(x)\beta(x) \quad . \tag{154}$$

We note that $\frac{1}{f(x)}$ has branch points at $x = \pm\,\gamma$ and $x = \pm\,1$ and simple poles at $x = \pm\,s$, where $s > 1$ is the location of a Rayleigh pole [14].

In view of these singularities the integrals (150) and (151) require interpreting. This interpretation must be in accordance with the radiation condition appropriate for an $e^{-i\omega t}$ time dependence. The contour of integration, in the complex z-plane, satisfying this requirement is shown in Figure 4, where we have taken the branch cuts outward along the real axis to infinity from $\pm\gamma$ and ± 1 for the functions $\alpha(z)$ and $\beta(z)$, respectively. We note that the radiation contour has infinitesimal indentations below the points γ, 1 and s, respectively. For later reference we have also included in (152) and (153) the appropriate definitions of $\alpha(x)$ and $\beta(x)$ on the negative real axis <u>below</u> the corresponding branch cuts.

We now use the expressions (150) and (151) to construct the fundamental singularities $\hat{U}_z^z(b,\rho)$ and $\hat{U}_r^z(b,\rho)$.

Referring to Figure 3, we have

$$\hat{U}_z^z(b,\rho) = -\frac{k}{8\pi\mu}\int_0^{2\pi}\int_0^{\infty}\frac{x\alpha(x)J_0(kxR)}{f(x)}\,dx\,d\varphi \tag{155}$$

where

$$R = \sqrt{b^2+\rho^2-2b\rho\cos(\theta-\varphi)} \quad . \tag{156}$$

Using an addition formula for Bessel functions [12], we can write

$$\hat{U}_z^z(b,\rho) = -\frac{k}{8\pi\mu}\int_0^{2\pi}\int_0^{\infty}\frac{x\alpha(x)}{f(x)}\sum_{m=-\infty}^{\infty} J_m(kbx)J_m(k\rho x)\cos m(\theta-\varphi)\,dx\,d\varphi \quad . \tag{157}$$

Changing the order of integration and integrating with respect to φ, we have

$$\hat{U}_z^z(b,\rho) = -\frac{k}{4\mu}\int_0^\infty \frac{x\alpha(x)}{f(x)} J_o(kbx)J_o(k\rho x)dx. \tag{158}$$

With the adoption of the notations

$$t = \frac{b}{a} \tag{159a}$$

and

$$u = \frac{\rho}{a} \tag{159b}$$

equation (158) can be written

$$\hat{U}_z^z(b,\rho) = -\frac{k}{4\mu}\int_0^\infty \frac{x\alpha(x)}{f(x)} J_o(t\hat{x})J_o(u\hat{x})dx \tag{160}$$

where $\hat{x}$ is given in (83).

To obtain the corresponding expression for $\hat{U}_r^z(b,\rho)$, we again refer to Figure 3. Using the notation indicated in this figure and the result (151) the displacement in the radial direction at P due to a unit force oscillating in the z-direction at Q is

$$\frac{k}{8\pi\mu}\int_0^\infty \frac{x^2[2x^2-1-2\alpha(x)\beta(x)]}{f(x)} J_1(kxR)dx \cdot \cos\psi \tag{161}$$

where R is given in (156). Hence

$$\hat{U}_r^z(b,\rho) = \frac{k}{8\pi\mu}\int_0^{2\pi}\int_0^\infty \frac{x^2[2x^2-1-2\alpha(x)\beta(x)]}{f(x)} J_1(kxR)\cos\psi \, dxd\varphi \tag{162}$$

Using an addition formula for Bessel functions we have

$$\hat{U}^{z}_{r}(b,\rho) = \tag{163}$$

$$\frac{k}{8\pi\mu}\int_{0}^{2\pi}\int_{0}^{\infty}\frac{x^2[2x^2-1-2\alpha(x)\beta(x)]}{f(x)}\sum_{m=-\infty}^{\infty}J_{m+1}(kbx)J_{m}(k\rho x)\cos[m(\theta-\varphi)]dxd\varphi \ .$$

Reversing the order of integration and integrating with respect to φ, we obtain

$$\hat{U}^{z}_{r}(b,\rho) = \frac{k}{4\mu}\int_{0}^{\infty}\frac{x^2[2x^2-1-2\alpha(x)\beta(x)]}{f(x)}J_{1}(kbx)J_{0}(k\rho x)dx \ . \tag{164}$$

If we use the notations given in (83), (159a) and (159b), we have

$$\hat{U}^{z}_{r}(b,\rho) = \frac{k}{4\mu}\int_{0}^{\infty}\frac{x^2[2x^2-1-2\alpha(x)\beta(x)]}{f(x)}J_{1}(t\hat{x})J_{0}(u\hat{x})dx \ . \tag{165}$$

APPENDIX B

METHODS OF ANALYSIS FOR INTEGRALS

We give examples to illustrate the methods used to reduce the integrals of the five types given in Chapter IV to forms suitable for numerical approximation.

As an example of Method I, we consider the integral

$$I_5 = \int_0^a \hat{U}_r^z(b,\rho)\,{}^1\tau_{zz}(\rho)\rho d\rho \tag{166}$$

$$= \frac{d}{\pi}\frac{\lambda+\mu}{\lambda+2\mu}(ka)\int_0^1 \left[\int_0^\infty \frac{x^2(2x^2-1-2\alpha(x)\beta(x))J_1(t\hat{x})J_o(u\hat{x})dx}{f(x)}\right]\frac{udu}{\sqrt{1-u^2}}$$

where ${}^1\tau_{zz}(\rho)$ and $\hat{U}_r^z(b,\rho)$ are as given in equations (20) and (165), respectively, and $u = \frac{\rho}{a}$. Using the identity [13]

$$\int_0^1 \frac{u\, J_o(u\hat{x})du}{\sqrt{1-u^2}} = \frac{\sin\hat{x}}{\hat{x}}\quad, \tag{167}$$

and the relationship (97), (166) leads to

$$I_5 = \frac{d}{\pi}(1-\gamma^2)\int_0^\infty \frac{x(2x^2-1-2\alpha(x)\beta(x))J_1(t\hat{x})\sin\hat{x}}{f(x)}dx\ . \tag{168}$$

Introducing the Hankel functions, we have

$$I_5 = \frac{d}{2\pi}(1-\gamma^2)\Big[\int_0^\infty \frac{x(2x^2-1-2\alpha(x)\beta(x))H_1^{(1)}(t\hat{x})\sin\hat{x}}{f(x)}\,dx \tag{169}$$

$$+\int_0^\infty \frac{x(2x^2-1-2\alpha(x)\beta(x))H_1^{(2)}(t\hat{x})\sin\hat{x}}{f(x)}\,dx\Big].$$

The integrals in (169) are understood to be taken along the radiation contour described in Appendix A.

To reduce (169) to a sum of finite integrals we consider the contour integral

$$\int_\Gamma \frac{z[2z^2-1-2\alpha(z)\beta(z)]H_1^{(2)}(t\hat{z})\sin\hat{z}}{f(z)}\,dz\,, \tag{170}$$

where $\hat{z} = kaz$ and Γ is the contour shown in Figure 5. We note that $\alpha(z)$ and $\beta(z)$ as defined in (152) and (153) have been extended analytically into the lower half-plane with branch cuts outward along the real axis to ∞ for $\pm\gamma$ and ± 1 for $\alpha(z)$ and $\beta(z)$, respectively. The contour Γ has semicircular indentations beneath the branch points and the simple poles $\pm s$. There is also a branch cut along the negative real axis for the function $H_1^{(2)}(t\hat{z})$.

By Cauchy's Theorem the integral (170) is zero. Also, since [15]

$$\frac{z[2z^2-1-2\alpha(z)\beta(z)]H_1^{(2)}(t\hat{z})\sin\hat{z}}{f(z)} \tag{171}$$

$$\sim \frac{M_1}{z^{3/2}}\left[e^{-i(t-1)\hat{z}} - e^{i(t+1)\hat{z}}\right],\ z\to\infty,\ \mathrm{Im}(z)<0\,,$$

where M_1 is a complex constant and because $t > 1$, Jordan's Lemma applies, and therefore

$$\lim_{T\to\infty} \int_{\Gamma_T} \frac{z[2z^2-1-2\alpha(z)\beta(z)]H_1^{(2)}(t\hat{z})\sin \hat{z}}{f(z)}\, dz = 0 \,. \tag{172}$$

Letting the radii, ϵ, ϵ_1, ϵ_2 and ϵ_3, of the semicircular indentations tend to zero, we obtain

$$\begin{aligned} &⨍_{-\infty}^{1} T_2(x)dx + \int_{-1}^{-\gamma} T_2(x)dx + \int_{-\gamma}^{0} T_2(x)dx + R(-s) \\ &+ \int_{0}^{\gamma} T_2(x)dx + \int_{\gamma}^{1} T_2(x)dx + ⨍_{1}^{\infty} T_2(x)dx + R(s) = 0 \,, \end{aligned} \tag{173}$$

where $R(s)$ and $R(-s)$ are the contributions from the Rayleigh poles, s and -s, respectively, and

$$T_2(x) = \frac{x[2x^2-1-2\alpha(x)\beta(x)]H_1^{(2)}(t\hat{x})\sin \hat{x}}{f(x)} \,. \tag{174}$$

Here and in subsequent expressions the notations $⨍_{1}^{\infty}$ and $⨍_{-\infty}^{-1}$ denote the Cauchy principle value integrals with respect to the poles s and -s, respectively. Hence

$$\begin{aligned} &⨍_{1}^{\infty} T_2(x)dx + \int_{0}^{\gamma} T_2(x)dx + \int_{\gamma}^{1} T_2(x)dx + R(s) \\ &= - ⨍_{-\infty}^{-1} T_2(x)dx - \int_{-1}^{-\gamma} T_2(x)dx - \int_{-\gamma}^{0} T_2(x)dx - R(-s) \end{aligned} \tag{175}$$

$$= - \int_1^{\infty} T_2(-x)dx - \int_{\gamma}^{1} T_2(-x)dx - \int_0^{\gamma} T_2(-x)dx \quad R(s) ,$$

where it has been noted that

$$R(-s) = \pi i \frac{-s[2s^2-1-2\alpha(-s)\beta(-s)]H_1^{(2)}(-t\hat{s})\sin(-\hat{s})}{f'(-s)} \tag{176}$$

$$= -\pi i \frac{s[2s^2-1-2\alpha(s)\beta(s)]H_1^{(1)}(t\hat{s})\sin \hat{s}}{f'(s)} = - R(s) ,$$

since $f'(-s) = f'(s)$ and $H_1^{(2)}(-t\hat{s}) = H_1^{(1)}(t\hat{s})$. Now, since $f'(s)$ is the same as $f_1(s)$ in (84), combining the results (169), (174), (175) and (176) and using the notation in (90), we obtain

$$I_5 = \frac{d}{\pi} (1-\gamma^2)\{-i \int_{\gamma}^{1} \hat{K}(x)H_1^{(1)}(t\hat{x})\sin \hat{x}\, dx \tag{177}$$

$$+ \frac{\pi i \; s[2s^2-1-2\sqrt{s^2-\gamma^2}\,\sqrt{s^2-1}]H_1^{(1)}(t\hat{s})\sin \hat{s}}{f_1(s)} ,$$

which is identical to equation (99).

Method 2 is illustrated by the integral

$$I_7 = \int_0^a \hat{U}_z^z(b,\rho)\overset{1}{\tau}_{zz}(\rho)\rho d\rho \tag{178}$$

$$= - \frac{d}{\pi}(1-\gamma^2)ak\int_0^1 \int_0^{\infty} \frac{x\alpha(x)}{f(x)} \; J_o(t\hat{x})J_o(u\hat{x}) \frac{u}{\sqrt{1-u^2}} dxdu$$

where equations (20), (160) and (159b) have been used. Changing the order of integration and using (167), we have

$$I_7 = -\frac{d}{\pi}(1-\gamma^2)\int_0^\infty \frac{\alpha(x)}{f(x)} J_o(t\hat{x}) \sin \hat{x}\, dx . \tag{179}$$

Introducing Hankel functions, we obtain

$$I_7 = -\frac{d}{2\pi}(1-\gamma^2)\left[\int_0^\infty \frac{\alpha(x)}{f(x)} H_o^{(1)}(t\hat{x}) \sin \hat{x}\, dx \right. \tag{180}$$

$$\left. + \int_0^\infty \frac{\alpha(x)}{f(x)} H_o^{(2)}(t\hat{x}) \sin \hat{x}\, dx \right]$$

where the integrals are taken over the radiation contour as previously explained in Appendix A. In order to reduce (180) to a form more suitable for numerical analysis, we consider the contour integral

$$\int_\Gamma \frac{\alpha(z)}{f(z)} H_o^{(2)}(t\hat{z}) \sin \hat{z}\, dz \tag{181}$$

where $\hat{z} = kaz$ and Γ is the contour shown in Figure 5. Branch cuts for $\alpha(z)$ and $\beta(z)$ are the same as those for the contour integral (170) with a branch cut taken along the negative real axis for the function $H_o^{(2)}(t\hat{z})$; $\alpha(z)$ and $\beta(z)$ have been extended analytically into the lower half-plane.

By Cauchy's Theorem the integral (181) is zero. Furthermore, since [15]

$$\frac{\alpha(z)H_o^{(2)}(t\hat{z})\sin\hat{z}}{f(z)} \sim \frac{M_2}{z^{3/2}}\left[e^{-i(t-1)\hat{z}} - e^{-i(t+1)\hat{z}}\right], \tag{182}$$

$$z \to \infty,\ \mathrm{Im}(z) < 0\ ,\quad t > 1\ ,$$

where M_2 is a complex constant, Jordan's Lemma applies, and therefore

$$\lim_{T\to\infty}\int_{\Gamma_T} \frac{\alpha(z)H_o^{(2)}(t\hat{z})\sin\hat{z}}{f(z)}\,dz = 0 \tag{183}$$

where Γ_T is shown in Figure 5. Hence, upon letting the radii, ϵ, ϵ_1, ϵ_2 and ϵ_3, of the indentations tend to 0,

$$\int_{-\infty}^{-1} T_3(x)dx + \int_{-1}^{-\gamma} T_3(x)dx + \int_{-\gamma}^{o} T_3(x)dx + R_1(-s) \tag{184}$$

$$+ \int_{o}^{\gamma} T_3(x)dx + \int_{\gamma}^{1} T_3(x)dx + \int_{1}^{\infty} T_3(x)dx + R_1(s) = 0\ ,$$

where

$$T_3(x) = \frac{\alpha(x)H_o^{(2)}(t\hat{x})\sin\hat{x}}{f(x)}, \tag{185}$$

and $R_1(-s)$ and $R_1(s)$ are the contributions from the Rayleigh poles, $-s$ and s, respectively. Hence

$$\int_0^{\gamma} T_3(x)dx + \int_{\gamma}^{1} T_3(x)dx + \int_1^{\infty} T_3(x)dx + R_1(s) \tag{186}$$

$$= - \int_{-\infty}^{-1} T_3(x)dx - \int_{-1}^{-\gamma} T_3(x)dx - \int_{-\gamma}^{0} T_3(x)dx - R_1(-s)$$

$$= -\int_1^{\infty} T_3(-x)dx - \int_{\gamma}^{1} T_3(-x)dx - \int_0^{\gamma} T_3(-x)dx - R_1(s)$$

where it has been observed that

$$R_1(-s) = \pi i \frac{-s\sqrt{s^2-\gamma^2}\, H_o^{(2)}(-t\hat{s})\sin(-\hat{s})}{f'(-s)} \tag{187}$$

$$= \pi i \frac{s\sqrt{s^2-\gamma^2}\, H_o^{(1)}(t\hat{s})\sin \hat{s}}{f'(s)} = R_1(s) \quad ,$$

since $f'(-s) = - f'(s)$ and $H_o^{(2)}(-t\hat{s}) = - H_o^{(1)}(t\hat{s})$. Combining the results (180), (185), (186) and (187), we obtain

$$I_7 = - \frac{d}{\pi} (1-\gamma^2)\Big[\int_{\gamma}^{1} H_o^{(1)}(t\hat{x})\sin \hat{x} \frac{\left(x^2 - \frac{1}{2}\right)^2 \sqrt{x^2-\gamma^2}}{\left(x^2 - \frac{1}{2}\right)^4 + x^4(x^2-\gamma^2)(1-x^2)}\, dx \tag{188}$$

$$+ \int_1^{\infty} H_o^{(1)}(t\hat{x})\sin \hat{x} \frac{\sqrt{x^2-\gamma^2}}{\left(x^2 - \frac{1}{2}\right)^2 - x^2\sqrt{x^2-\gamma^2}\,\sqrt{x^2-1}}\, dx\Big] .$$

The Cauchy principle value integral in (188) is not in a form convenient for numerical approximation. In order to obtain a more satisfactory expression, we consider the contour integral

$$\int_{\Gamma_1} \frac{\hat{M}(z)}{z} H_0^{(2)}(t\hat{z}) \sin \hat{z} \, dz \tag{189}$$

where

$$\hat{M}(z) = \frac{z\alpha_2(z)}{\left(z^2 - \frac{1}{2}\right)^2 - z^2\alpha_2(z)\beta_2(z)} , \tag{190}$$

Γ_1 is the contour shown in Figure 6, $\alpha_2(z)$ and $\beta_2(z)$ are the analytic continuations into the upper right quarter plane of the functions whose values are

$$\alpha_2(x) = \begin{cases} i\sqrt{\gamma^2 - x^2} , & 0 \le x \le \gamma \\ \sqrt{x^2 - \gamma^2} , & x > \gamma \end{cases} \tag{191}$$

and

$$\beta_2(x) = \begin{cases} i\sqrt{1 - x^2} , & 0 \le x \le 1 \\ \sqrt{x^2 - 1} , & x > 1 \end{cases} \tag{192}$$

on the positive real line with branch cuts on the real line outward to ∞ from γ and 1 for $\alpha_2(z)$ and $\beta_2(z)$, respectively. Γ_1 has semicircular indentations above the branch points γ and 1 and the pole s, respectively. The contour integral (189) is zero by Cauchy's Theorem. Also, since [15]

$$\frac{\hat{M}(z)}{z} H_o^{(1)}(t\hat{z})\sin \hat{z} \sim \frac{M_3}{z^{3/2}} [e^{i(t+1)\hat{z}} - e^{i(t-1)\hat{z}}] , \tag{193}$$

$$z \to \infty, t > 1, Im(z) > 0,$$

where M_3 is a complex constant, Jordan's lemma applies on Γ_{1T} as $T \to \infty$ (see Figure 6), and therefore

$$\lim_{T\to\infty} \int_{\Gamma_{1T}} \frac{\hat{M}(z)}{z} H_o^{(1)}(t\hat{z})\sin \hat{z}\, dz = 0 .$$

Therefore, upon letting $T \to \infty$ and ϵ_4, ϵ_5 and ϵ_6 tend to 0, we obtain

$$\begin{aligned} &\int_o^{\gamma} \frac{\hat{M}(x)}{x} H_o^{(1)}(t\hat{x})\sin \hat{x}\, dx + \int_{\gamma}^{1} \frac{\hat{M}(x)}{x} H_o^{(1)}(t\hat{x})\sin \hat{x}\, dx \\ &\quad + ⨍_1^{\infty} \frac{\hat{M}(x)}{x} H_o^{(1)}(t\hat{x})\sin \hat{x}\, dx + R_3(s) + \int_o^{i\infty} \frac{\hat{M}(z)}{z} H_o^{(1)}(t\hat{z})\sin \hat{z}\, dz \\ &\quad = 0 \end{aligned} \tag{194}$$

where $R_3(s)$ is the contribution from the Rayleigh pole s and $\int_o^{i\infty}$ indicates an integral on the imaginary axis. Solving for the Cauchy principle value integral in (194) and using the identity [15]

$$H_o^{(1)}(z\ e^{i\pi/2}) = \frac{2}{\pi i} K_o(z),\ - \pi < \arg z < \frac{\pi}{2} \ , \tag{195}$$

we have

$$\begin{aligned} &⨍_1^{\infty} \frac{\hat{M}(x)}{x} H_o^{(1)}(t\hat{x})\sin \hat{x}\, dx \\ &\quad = - \int_o^{\gamma} \frac{\hat{M}(x)}{x} H_o^{(1)}(t\hat{x})\sin \hat{x}\, dx - \int_{\gamma}^{1} \frac{\hat{M}(x)}{x} H_o^{(1)}(t\hat{x})\sin \hat{x}\, dx \end{aligned} \tag{196}$$

$$+ \pi i \frac{\alpha_2(s) H_o^{(1)}(t\hat{s}) \sin \hat{s}}{f'(s)} - \frac{2}{\pi} \int_o^\infty \frac{\sqrt{x^2+\gamma^2}\, K_o(t\hat{x}) \sinh \hat{x}\, dx}{f_2(x)}$$

where $f_2(x)$ is given in (85). Now, since $f'(s)$ is the same as $f_1(s)$ in (84), combining results (188) and (196) we have

$$I_7 = -\frac{d}{\pi}(1-\gamma^2)\Big[-\frac{2}{\pi}\int_o^\infty \frac{\sqrt{x^2+\gamma^2}\, K_o(t\hat{x}) \sinh \hat{x}\, dx}{f_2(x)} \tag{197}$$

$$- i \int_o^\gamma \frac{K(x)}{x} H_o^{(1)}(t\hat{x}) \sin \hat{x}\, dx - i \int_\gamma^1 \frac{K(x)}{x} H_o^{(1)}(t\hat{x}) \sin \hat{x}\, dx$$

$$+ i\, \pi \frac{\sqrt{s^2-\gamma^2}\, H_o^{(1)}(t\hat{s}) \sin \hat{s}}{f_1(s)}\Big]$$

where $K(x)$ is given in (87). We note that this agrees with (101).

As an illustration of Method 3, we consider the integral

$$I_9 = \int_o^a \int_o^a \overset{1*r}{\tau}_{zz}(b,\rho')\hat{U}_z^z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho\rho' d\rho d\rho' \tag{198}$$

$$= \frac{2}{\pi^2}\gamma^2(1-\gamma^2)\frac{da^2k}{b}\int_o^1\int_o^1\int_o^\infty \frac{u}{\sqrt{1-u^2}}\frac{v}{\sqrt{1-v^2}}\frac{x\alpha(x)}{f(x)} J_o(u\hat{x})J_o(v\hat{s})\,dx\,du\,dv$$

where we have used (20), (46) and (160) and where $u = \frac{\rho}{a}$ and $v = \frac{\rho'}{a}$. Again the infinite integral is interpreted as in Appendix A. Reversing the order of integration we have

$$I_9 = \frac{2}{\pi^2} \gamma^2(1-\gamma^2) \frac{da^2k}{b} \int_0^\infty \frac{x\alpha(x)}{f(x)} \left[\int_0^1 \frac{uJ_o(u\hat{x})du}{\sqrt{1-u^2}}\right]^2 dx \tag{199}$$

where the symmetry with respect to u and v has been employed. From the identity (167) it follows that

$$I_9 = \frac{2}{\pi^2} \gamma^2(1-\gamma^2) \frac{da^2k}{b} \int_0^\infty \frac{x\alpha(x)}{f(x)} \left[\frac{\sin \hat{x}}{\hat{x}}\right]^2 dx \tag{200}$$

$$= \frac{1}{\pi^2} \gamma^2(1-\gamma^2) \frac{d}{t} \int_0^\infty \frac{\alpha(x)}{f(x)} \frac{1 - \cos 2\hat{x}}{\hat{x}} dx$$

$$= \frac{1}{2\pi^2} \gamma^2(1-\gamma^2) \frac{d}{t} \left[\int_0^\infty \frac{\alpha(x)}{f(x)} \frac{1 - e^{2i\hat{x}}}{\hat{x}} dx + \int_0^\infty \frac{\alpha(x)}{f(x)} \frac{1-e^{-2i\hat{x}}}{\hat{x}} dx\right]$$

where $t = \frac{b}{a}$. To reduce the final expression in (200) to a sum of finite integrals we consider the contour integral

$$\int_\Gamma \frac{\alpha(z)}{f(z)} \frac{1-e^{-2i\hat{z}}}{\hat{z}} dz \tag{201}$$

where Γ is the contour shown in Figure 5, and branch cuts are taken as in the treatment of the contour integral (170) for the functions $\alpha(z)$ and $\beta(z)$. Using a procedure similar to the one used for I_5 we obtain

$$I_9 = - \frac{i}{\pi^2} \gamma^2(1-\gamma^2) \frac{d}{t} \left[\int_0^1 \frac{K(x)}{x} \frac{1-e^{2i\hat{x}}}{\hat{x}} dx \right. \tag{202}$$

$$\left. - \pi \frac{\alpha(s)}{f_1(s)} \frac{1-e^{2i\hat{s}}}{\hat{s}}\right]$$

where we note that $f_1(s) = f'(s)$. Since

$$T_1(x) = \frac{1-e^{2i\hat{x}}}{\hat{x}} \ ,$$

this agrees with (103).

We now consider the integral

$$I_{10} = \int_0^a \int_0^a {}^{1*r}_{\tau_{zz}}(b,\rho')\hat{U}^z_z(\rho',\rho)\,{}^1_{\tau_{zz}}(\rho)\rho^3\rho'^3 d\rho d\rho' \tag{203}$$

$$= \frac{2}{\pi^2} \gamma^2(1-\gamma^2) \frac{da^6k}{6} \int_0^1\int_0^1\int_0^\infty \frac{u^3}{\sqrt{1-u^2}} \frac{v^3}{\sqrt{1-v^2}} \frac{x\alpha(x)J_o(u\hat{x})J_o(v\hat{x})}{f(x)} dxdudv \ ,$$

where we have used (20), (46) and (160) and where $u = \frac{\rho}{a}$ and $v = \frac{\rho'}{a}$, to illustrate Method 4. Interchanging the order of integration and using the symmetry in u and v we have

$$I_{10} = \frac{2}{\pi^2} \gamma^2(1-\gamma^2) \frac{da^6k}{b} \int_0^\infty \frac{x\alpha(x)}{f(x)} \left[\int_0^1 \frac{u^3J_o(u\hat{x})du}{\sqrt{1-u^2}} \right]^2 dx \ . \tag{204}$$

Using the identity

$$\int_0^1 \frac{u^3J_o(u\hat{x})du}{\sqrt{1-u^2}} = \left(\frac{\pi}{2}\right)^{1/2} \left[\frac{J_{1/2}(\hat{x})}{\hat{x}^{1/2}} - \frac{J_{3/2}(\hat{x})}{\hat{x}^{3/2}}\right] \ , \tag{205}$$

which can be derived by making the transformation u = sin t, referring to [18] and using the identity [15]

$$J_{5/2}(z) = \frac{3}{z} J_{3/2}(z) - J_{1/2}(z) \ ,$$

(204) becomes

$$I_{10} = \frac{1}{\pi}\, \gamma^2(1-\gamma^2)\, \frac{da^5k}{b} \Big[\int_0^\infty M(x)\, \frac{J_{1/2}{}^2(\hat{x})}{\hat{x}}\, dx \tag{206}$$

$$- 2\int_0^\infty M(x)\, \frac{J_{1/2}(\hat{x})J_{3/2}(\hat{x})}{\hat{x}^2}\, dx + \int_0^\infty M(x)\, \frac{J_{3/2}{}^2(\hat{x})}{\hat{x}^3}\, dx\Big]$$

where

$$M(x) = \frac{x\alpha(x)}{f(x)} \quad . \tag{207}$$

Since by [15]

$$J_{1/2}\,(\hat{x}) = \left(\frac{2}{\pi\hat{x}}\right)^{1/2} \sin\hat{x}\ , \tag{208}$$

$$\int_0^\infty \frac{x\alpha(x)}{f(x)}\, \frac{J_{1/2}{}^2(\hat{x})}{\hat{x}}\, dx = \frac{2}{\pi}\int_0^\infty \frac{x\alpha(x)}{f(x)} \left(\frac{\sin\hat{x}}{\hat{x}}\right)^2 dx \tag{209}$$

and

$$\int_0^\infty \frac{x\alpha(x)}{f(x)}\, \frac{J_{1/2}(\hat{x})J_{3/2}(\hat{x})}{\hat{x}^2}\, dx$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \frac{x\alpha(x)}{f(x)}\, \frac{J_{3/2}(\hat{x})\sin\hat{x}}{\hat{x}^{5/2}}\, dx \quad . \tag{210}$$

We observe that the integral (209) has already been considered in connection with I_9.

The integral

$$I'_{10} = \int_0^\infty M(x)\ J_{3/2}\ (\hat{x})\hat{x}^{-5/2}\ \sin \hat{x}\ dx \tag{211}$$

was treated by first using the integral representation [15]

$$J_{3/2}(\hat{x}) = \frac{\hat{x}^{3/2}}{\sqrt{2\pi}} \int_0^1 (1-v^2)\cos(v\hat{x})dv \tag{212}$$

obtaining

$$I'_{10} = \frac{1}{\sqrt{2\pi}} \int_0^1 (1-v^2) \left[\int_0^\infty M(x)\ \frac{\sin \hat{x}\ \cos v\hat{x}}{\hat{x}}\ dx\right] dv\ . \tag{213}$$

The integral

$$\int_0^\infty M(x)\ \frac{\sin \hat{x}\ \cos v\hat{x}}{\hat{x}}\ dx \tag{214}$$

$$= \frac{1}{4i} \left\{ \int_0^\infty M(x)\left[\frac{e^{i(1+v)\hat{x}} + e^{i(1-v)\hat{x}}}{\hat{x}}\right] dx \right.$$

$$\left. - \int_0^\infty M(x)\ \left[\frac{e^{-i(1+v)\hat{x}} - e^{-i(1-v)\hat{x}}}{\hat{x}}\right] dx \right\} ,\quad v < 1\ ,$$

can be reduced to a sum of finite integrals by considering the contour integral

$$\int_\Gamma M(z)\ \left[\ \frac{e^{-i(1+v)\hat{z}} + e^{-i(1-v)\hat{z}}}{\hat{z}}\right] dz \tag{215}$$

where Γ is the contour in Figure 5. We refer to the explanation following the contour integral (170) for the necessary branch cuts for the functions $\alpha(z)$ and $\beta(z)$. The procedure is similar to the one used for the integral I_5, and the result is

$$\int_0^\infty M(x)\,\frac{\sin\hat{x}\cos v\hat{x}}{\hat{x}}\,dx = \frac{1}{2ka}\left\{\int_0^1 \frac{K(x)}{x}\,T_4(v,\hat{x})dx - \frac{\pi\sqrt{s^2-\gamma^2}}{f'(s)}\,T_4(v,\hat{s})\right\} \tag{216}$$

where $K(x)$ is given in (87) and

$$T_4(v,\hat{x}) = e^{i(1+v)\hat{x}} + e^{i(1-v)\hat{x}} \quad . \tag{217}$$

Using the result (213), reversing the order of integration, and integrating from 0 to 1 with respect to v, we obtain

$$I'_{10} = -\sqrt{\frac{2}{\pi}}\,\frac{1}{ka}\left\{\int_0^1 \frac{K(x)}{x}\,e^{i\hat{x}}\left[\frac{\sin\hat{x}}{\hat{x}^3} - \frac{\cos\hat{x}}{\hat{x}^2}\right]dx - \pi\,\frac{\sqrt{s^2-\gamma^2}}{f'(s)}\,e^{i\hat{s}}\left[\frac{\sin\hat{s}}{\hat{s}^3} - \frac{\cos\hat{s}}{\hat{s}^2}\right]\right\} \quad . \tag{218}$$

The third integral in (206) is

$$\begin{aligned} I''_{10} &= \int_0^\infty M(x)\,\frac{J_{3/2}{}^2(\hat{x})}{\hat{x}^3}\,dx \\ &= \int_0^\infty \left[M(x) + \frac{2}{1-\gamma^2}\right]\frac{J_{3/2}{}^2(\hat{x})}{\hat{x}^3}\,dx - \frac{2}{1-\gamma^2}\int_0^\infty \frac{J_{3/2}{}^2(\hat{x})}{\hat{x}^3}\,dx \;. \end{aligned} \tag{219}$$

From [12] we obtain

$$\int_0^\infty \frac{J_{3/2}{}^2(\hat{x})}{\hat{x}^3}\,dx = \frac{2}{15}\,\frac{1}{ka} \quad . \tag{220}$$

Since by [15]

$$J_{3/2}(\hat{x}) = \sqrt{\frac{2}{\pi}}\left[\frac{\hat{x}\cos\hat{x} - \sin\hat{x}}{\hat{x}^{3/2}}\right], \tag{221}$$

we have

$$I_{10}''' = \int_0^\infty \left[M(x) + \frac{2}{1-\gamma^2}\right]\frac{J_{3/2}{}^2(\hat{x})}{\hat{x}^3}\,dx \tag{222}$$

$$= \frac{2}{\pi}\int_0^\infty \left[M(x) + \frac{2}{1-\gamma^2}\right]\left[\frac{\hat{x}^2\cos^2\hat{x} - 2\hat{x}\sin\hat{x}\cos\hat{x} + \sin^2\hat{x}}{\hat{x}^6}\right]dx \ .$$

The use of several trigonometric identities yields

$$I_{10}''' = \frac{1}{\pi}\left\{\int_0^\infty \left[M(x) + \frac{2}{1-\gamma^2}\right]\left[\frac{\hat{x}^2+1+\frac{2}{3}i\hat{x}^3+\frac{4}{15}i\hat{x}^5+e^{2i\hat{x}}(\hat{x}^2-2i\hat{x}+1)}{\hat{x}^6}\right]dx \right. \tag{223}$$

$$\left. + \int_0^\infty \left[M(x) + \frac{2}{1-\gamma^2}\right]\left[\frac{\hat{x}^2+1-\frac{2}{3}i\hat{x}^3-\frac{4}{15}i\hat{x}^5+e^{-2i\hat{x}}(\hat{x}^2-2i\hat{x}-1)}{\hat{x}^6}\right]dx\right\} \ .$$

The integral (222) was broken up in this manner in order to preserve convergence. To reduce (223) to a sum of finite integrals we consider the contour integral

$$\int_{\Gamma}\left[M(z)+\frac{2}{1-\gamma^2}\right]\left[\frac{1+\hat{z}^2-\frac{2}{3}i\hat{z}^3-\frac{4}{15}i\hat{z}^5+e^{-2i\hat{z}}(\hat{z}^2-2i\hat{z}-1)}{\hat{z}^6}\right]dz \quad (224)$$

where Γ is the contour shown in Figure 5. For associated branch cuts we refer to the explanation following the contour integral (170). Since

$$M(z)+\frac{2}{1-\gamma^2}\sim\frac{M_4}{z^2}\,,\quad z\to\infty,\ \operatorname{Im}(z)<0\quad, \qquad (225)$$

where M_4 is a complex constant,

$$\lim_{T\to\infty}\int_{\Gamma_T}\left[M(z)+\frac{2}{1-\gamma^2}\right]\left[\frac{1+\hat{z}^2-\frac{2}{3}i\hat{z}^3-\frac{4}{15}i\hat{z}^5+e^{-2i\hat{z}}(\hat{z}^2-2i\hat{z}-1)}{\hat{z}^6}\right]dz \quad (226)$$
$$=0\,.$$

By a procedure similar to the one used for the integral I_5, we obtain

$$I_{10}^{*}=-\frac{2i}{\pi}\left\{\int_0^{\gamma}K(x)q(\hat{x})dx+\int_{\gamma}^{1}K(x)q(\hat{x})dx-\frac{\pi s\sqrt{s^2-\gamma^2}}{f'(s)}\,q'(\hat{s})\right\} \qquad (227)$$

where $K(x)$ and $q(\hat{x})$ are given in equations (87) and (88), respectively.

Finally, combining results (206), (218), (219), (220) and (227), we have (see equation (89))

$$I_{10}=\frac{2}{\pi^2}\gamma^2(1-\gamma^2)\,\frac{da^5}{b}\left[\int_0^1 K(x)\,\frac{Q(\hat{x})}{x}\,dx-\frac{\pi\sqrt{s^2-\gamma^2}\,Q(\hat{s})}{f_1(s)}-\frac{2\pi}{15}\,\frac{1}{1-\gamma^2}\right], \qquad (228)$$

where it has been noted that $f_1(s) = f'(s)$ and $t = b/a$. This result agrees with (104).

The integral I_{11} will be used to illustrate Method 5. Upon using (20), (35) and (160), we have

$$I_{11} = \int_0^a \int_0^a \overset{1*z}{\tau}_{zz}(b,\rho')\hat{U}_z^z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho\rho' d\rho d\rho' \tag{229}$$

$$= -\frac{2dak}{\pi^2}(1-\gamma^2\sqrt{t^2-1}\int_0^1\int_0^1 \frac{u}{\sqrt{1-u^2}\,[t^2-u^2]}$$

$$\left[\int_0^\infty M(x)J_o(u\hat{x})J_o(v\hat{x})dx\right]\frac{vdudv}{\sqrt{1-v^2}}$$

where $M(x)$ is given in (207). Changing the order of integration and using the identity (167) we have

$$I_{11} = -\frac{2d}{\pi^2}(1-\gamma^2)\sqrt{t^2-1}\,(ka)\int_0^1 \frac{u}{\sqrt{1-u^2}\,[t^2-u^2]} \tag{230}$$

$$\left[\int_0^\infty M(x)J_o(u\hat{x})\frac{\sin\hat{x}}{\hat{x}}dx\right]du \quad .$$

We proceed to reduce

$$I_{11}' = \int_0^\infty M(x)\ J_o(u\hat{x})\ \frac{\sin\hat{x}}{\hat{x}}dx\ ,\ 0 \le u \le 1\ , \tag{231}$$

to a sum of finite integrals. We first note that

$$I_{11}' = \left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty M(x)J_o(u\hat{x})J_{1/2}(\hat{x})\hat{x}^{-1/2}\,dx \tag{232}$$

$$= \left(\frac{\pi}{2}\right)^{1/2} \left\{ \int_0^\infty \left[M(x) + \frac{2}{1-\gamma^2} \right] J_o(u\hat{x}) J_{1/2}(\hat{x}) \hat{x}^{-1/2} \, dx \right.$$

$$\left. - \frac{2}{1-\gamma^2} \int_0^\infty J_o(u\hat{x}) J_{1/2}(\hat{x}) \hat{x}^{-1/2} \, dx \right\} .$$

From [12] we obtain

$$\int_0^\infty J_o(u\hat{x}) J_{1/2}(\hat{x}) \hat{x}^{-1/2} \, dx = \left(\frac{\pi}{2}\right)^{1/2} \frac{1}{ak} . \tag{233}$$

To treat the integral

$$I''_{11} = \int_0^\infty \left[M(x) + \frac{2}{1-\gamma^2} \right] J_o(u\hat{x}) J_{1/2}(\hat{x}) \hat{x}^{-1/2} \, dx , \tag{234}$$

we use the integral representation [15]

$$J_o(u\hat{x}) = \frac{2}{\pi} \int_0^1 \frac{\cos(\eta u\hat{x}) d\eta}{\sqrt{1-\eta^2}} , \tag{235}$$

and

$$J_{1/2}(\hat{x}) = \left(\frac{2\hat{x}}{\pi}\right)^{1/2} \int_0^1 \cos(w\hat{x}) dw , \tag{236}$$

to obtain, after changing the order of integration,

$$I'''_{11} = \left(\frac{2}{\pi}\right)^{3/2} \int_0^1 \int_0^1 \int_0^\infty \frac{1}{\sqrt{1-\eta^2}} \left[M(x) + \frac{2}{1-\gamma^2} \right] \cos(\eta u\hat{x}) . \tag{237}$$

$$\cos(w\hat{x}) dx dw d\eta .$$

The integral

$$I'''_{11} = \int_0^\infty \left[M(x) + \frac{2}{1-\gamma^2}\right] \cos(u\eta\hat{x}) \cos(w\hat{x})dx \tag{238}$$

$$= \frac{1}{4}\left\{ \int_0^\infty \left[M(x) + \frac{2}{1-\gamma^2}\right]\left[e^{i(u\eta+w)\hat{x}} + e^{i|u\eta-w|\hat{x}}\right] dx \right.$$

$$\left. + \int_0^\infty \left[M(x) + \frac{2}{1-\gamma^2}\right]\left[e^{-i(u\eta+w)\hat{x}} + e^{-i|u\eta-w|\hat{x}}\right] dx \right\}$$

can be reduced to a sum of finite integrals by considering the contour integral

$$\int_\Gamma \left[M(z) + \frac{2}{1-\gamma^2}\right]\left[e^{-i(u\eta+w)\hat{z}} + e^{-i|u\eta-w|\hat{z}}\right] dz \; , \tag{239}$$

$$u\eta \neq w \; ,$$

where branch cuts are taken as in the contour integral (170) for $\alpha(z)$ and $\beta(z)$ (see Figure 5). The result is

$$I'''_{11} = -\frac{i}{2}\left[\int_0^1 K(x)R(\hat{x},\eta,w,u)dx - \pi \frac{s\sqrt{s^2-\gamma^2}}{f'(s)} R(\hat{s},\eta,w,u)\right] , \tag{240}$$

where

$$R(\hat{x},\eta,w,u) = e^{i(u\eta+w)\hat{x}} + e^{i|u\eta-w|\hat{x}} \; . \tag{241}$$

We now note that

$$\int_0^1 \int_0^1 \frac{R(\hat{x},\eta,w,u)}{\sqrt{1-\eta^2}} \, d\eta dw \tag{242}$$

$$= \int_0^1 \int_0^1 \frac{e^{i(u\eta+w)\hat{x}}}{\sqrt{1-\eta^2}} \, d\eta dw + \int_0^1 \int_0^{\eta t} \frac{e^{i\eta u\hat{x}} e^{-iw\hat{x}}}{\sqrt{1-\eta^2}} \, dw d\eta$$

$$+ \int_0^1 \int_{t\eta}^1 \frac{e^{-i\eta u\hat{x}} e^{iw\hat{x}}}{\sqrt{1-\eta^2}} \, dw d\eta$$

$$= \left[\frac{e^{i\hat{x}}-1}{i\hat{x}} \right] \left[\frac{\pi}{2} J_o(u\hat{x}) + i \frac{\pi}{2} H_o(u\hat{x}) \right] + \frac{i\pi}{\hat{x}}$$

$$+ \frac{1}{i\hat{x}} \frac{\pi}{2} [J_o(u\hat{x}) + i H_o(u\hat{x})] + \frac{e^{i\hat{x}}}{i\hat{x}} \left[\frac{\pi}{2} J_o(u\hat{x}) - i \frac{\pi}{2} H_o(u\hat{x}) \right]$$

$$= \frac{\pi}{i\hat{x}} [e^{i\hat{x}} J_o(u\hat{x}) - 1] \, ,$$

where $H_o(u\hat{x})$ is the Struve function with argument $u\hat{x}$. [19]

Combining (232), (233), (241) and (242), we have

$$I'_{11} = - \frac{i}{\pi} \left[\int_0^1 \frac{K(x) V(\hat{x},u)}{\hat{x}} \, d - \pi \frac{\sqrt{s^2-\gamma^2} \, V(\hat{s},u)}{f'(s)} \right] - \frac{\pi}{1-\gamma^2} \frac{1}{ak} \tag{243}$$

where

$$V(\hat{x},u) = i\pi[1 - e^{i\hat{x}} J_o(u\hat{x})] \, .$$

Finally

$$I_{11} = -\frac{2d}{\pi^2}(1-\gamma^2)\sqrt{t^2-1} \tag{244}$$

$$\cdot \int_0^1 \frac{u}{\sqrt{1-u^2}\,[t^2-u^2]} \left\{-\frac{i}{\pi}\left[\int_0^1 \frac{K(x)V(\hat{x},u)dx}{x} - \pi\frac{\sqrt{s^2-\gamma^2}\,V(\hat{s},u)}{f'(s)}\right]\right.$$

$$\left. - \frac{\pi}{1-\gamma^2}\right\} du \quad .$$

The integrals I_{14}, I_{15} and I_{16} are related to the integrals I_9, I_{11}, I_{12} and I_{13} as follows.

$$I_{14} = \int_0^a \int_0^a \overset{1*r}{\tau}_{zz}(\rho',b)\hat{U}_z^z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho\rho'^3 d\rho d\rho' \tag{245}$$

$$= A\int_0^1\int_0^1 \frac{u^3}{\sqrt{1-u^2}}\left[\int_0^\infty M(x)J_0(u\hat{x})J_0(v\hat{x})dx\right]\frac{v}{\sqrt{1-v^2}}\,dudv$$

$$= A\int_0^1\int_0^1 \frac{u}{\sqrt{1-u^2}}\left[\int_0^\infty M(x)J_0(u\hat{x})J_0(v\hat{x})dx\right]\frac{v^3}{\sqrt{1-v^2}}\,dudv$$

$$= \int_0^a \int_0^a \overset{1*r}{\tau}_{zz}(\rho',b)\hat{U}_z^z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho^3\rho' d\rho d\rho' = I_{13} \quad .$$

(Here A is a constant.)

$$I_{15} = \int_0^a \int_0^a \overset{1*z}{\tau}_{zz}(\rho',b)\hat{U}_z^z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho(\rho')^3 d\rho d\rho' \tag{246}$$

$$= -\frac{2}{\pi}\sqrt{b^2-a^2}\int_0^a\int_0^a \frac{\rho^3}{\sqrt{a^2-\rho^2}\,[b^2-\rho'^2]}\,\hat{U}_z^z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho d\rho d\rho'$$

$$= b^2\int_0^a\int_0^a \overset{1*z}{\tau}_{zz}(\rho',b)\hat{U}_z^z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho\rho' d\rho d\rho'$$

$$+\frac{a^2 b\sqrt{b^2-a^2}}{\gamma^2}\int_0^a\int_0^a \overset{1*r}{\tau}_{zz}(\rho',b)\hat{U}_z^z(\rho',\rho)\overset{1}{\tau}_{zz}(\rho)\rho\rho' d\rho d\rho'$$

$$= b^2 I_{11} + \frac{a^2 b\sqrt{b^2-a^2}}{\gamma^2} I_9 ,$$

which agrees with (109). The result (110) is derived in a similar manner.

APPENDIX C

SOME PROPERTIES OF THE ADMISSIBLE FUNCTIONS $\overset{a}{\tau}_{zz}(\rho)$, $\overset{a*z}{\tau}_{zz}(\rho,b)$ and $\overset{a*r}{\tau}_{zz}(\rho,b)$

The admissible functions $\overset{a}{\tau}_{zz}(\rho)$, $\overset{a*z}{\tau}_{zz}(\rho,b)$ and $\overset{a*r}{\tau}_{zz}(\rho,b)$ were constructed so that they possessed certain properties of the corresponding exact stress distributions.

It is well known that the exact stress distributions $\tau_{zz}(\rho)$, $\tau^{*r}_{zz}(\rho,b)$ and $\tau^{*z}_{zz}(\rho,b)$, $0 \le \rho < a, b > a$, each have a singularity of the form $\dfrac{1}{\sqrt{a^2-\rho^2}}$ at the edge of the disc.

We also claim that the these exact stress distributions are even functions of ρ, and we will now verify this.

In order to determine $\tau_{zz}(\rho)$, $0 \le \rho < a$, we must solve the boundary value problem (1a,b,c,d) modified for our axisymmetric problem. That is, we must consider the boundary conditions

$$\tau_{rz}(\rho) = 0 \ , \ \rho \ge 0 \tag{247}$$

$$u_z(\rho) = d \ , \ 0 \le \rho < a \tag{248}$$

and

$$\tau_{zz}(\rho) = 0, \ \rho > a \tag{249}$$

on $B = \{(x,y,z):z = 0\}$ where $\tau_{rz}(\rho)$ is the shear stress at $(\rho,\varphi,0)$. From [20] these conditions are equivalent to

$$- \mu k^3 \int_0^\infty [2A(kx)x^2 + C(kx)(2x^2 - 1)]J_1(\rho kx)dx = 0, \ \rho \geq 0 \ , \tag{250}$$

$$k^2 \int_0^\infty x[A(kx) + C(kx)]J_o(\rho kx)dx = d, \ 0 \leq \rho < a, \tag{251}$$

and

$$- \mu k^3 \int_0^\infty \left[\frac{A(kx)(2x^2-1)}{\alpha(x)} + 2\beta(x)C(kx)\right] J_o(\rho kx)dx = 0 \ , \ \rho > a \ , \tag{252}$$

respectively, for the functions $A(kx)$ and $C(kx)$. Equation (250) implies that

$$2A(kx)x^2 + C(kx)(2x^2 - 1) = 0 \ . \tag{253}$$

We obtain from equations (251), (252) and (253) the dual integral equations

$$k^2 \int_0^\infty \frac{x}{2x^2 - 1} J_o(\rho kx)A(kx)dx = d, \ 0 \leq \rho < a \ , \tag{254}$$

and

$$- \mu k^3 \int_0^\infty x \left[\frac{2x^2 - 1}{\alpha(x)} - \frac{4x^2\beta(x)}{2x^2 - 1}\right] J_o(\rho kx)A(kx)dx = 0, \ \rho > a \ . \tag{255}$$

We extract $\tau_{zz}(\rho)$ [20] as

$$\tau_{zz}(\rho) = - \mu k^3 \int_0^\infty x \left[\frac{2x^2 - 1}{\alpha(x)} + \frac{4x^2\beta(x)}{1 - 2x^2}\right] J_o(\rho kx) \ A(kx)dx \ . \tag{256}$$

Since $J_o(y)$ is an even function of its argument y, $\tau_{zz}(\rho)$ is an even function of ρ.

The function $\tau_{zz}^{*z}(\rho,b)$ is obtained in a similar manner by first considering the boundary value problem (1a,b,c,d) for axisymmetric problems with the associated axisymmetric boundary conditions

$$\tau_{rz}^{*z}(\rho,b) = 0, \ \rho \geq 0 , \tag{257}$$

$$\tau_{zz}^{*z}(\rho,b) = f(\rho), \ \rho > a , \tag{258}$$

and

$$u_z^{*z}(\rho,b) = 0 , \ 0 \leq \rho < a , \tag{259}$$

where $f(\rho)$ is a generalized function representing the stress distribution due to a ring of unit point forces applied at $\rho = b$ in the direction. By an analysis similar to the one given for $\tau_{zz}(\rho)$ we obtain

$$\tau_{zz}^{*z}(\rho,b) = -\mu k^3 \int_0^\infty x \left[\frac{2x^2 - 1}{\alpha(x)} + \frac{4x^2\beta(x)}{1 - 2x^2}\right] A_1(kx)J_0(\rho kx)dx , \tag{260}$$

where the function $A_1(kx)$ is determined by the boundary value problem. Hence again since $J_0(y)$ is an even function of its argument y, $\tau_{zz}^{*z}(\rho,b)$ is an even function of ρ. By a similar argument, we can also show that $\tau_{zz}^{*r}(\rho,b)$ is an even function of ρ.

APPENDIX D

COMPUTER PROGRAMS USED TO COMPUTE THE VARIATIONAL APPROXIMATIONS

```
      PROGRAM UZ20(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      COMPLEX R5,R6,RG,RH,RR1,RR2,RI,RJ,ANUM1,ANUM2,DEN
      COMPLEX VAR1,VAR2
C THIS PROGRAM COMPUTES THE FIRST AND SECOND VARIATIONAL
C APPROXIMATIONS FOR UZ
      G=.57735
      S=1.087663874
      READ *,N
      WRITE(6,87)
   87   FORMAT(*APPROXIMATION FOR UZ*)
      DO 310 K=1,N
      READ *,T,AK
      PI=3.1415926536
C COMPUTE THE INTEGRAL -I7
      CALL S5(AK,T,G,S,R5)
C COMPUTE THE INTEGRAL -I11
      CALL S6(AK,T,G,S,R6)
C COMPUTE THW INTEGRAL I3
      R7=2.*ASIN(1./T)/PI
C COMPUTE THE INTEGRAL I4
      RF=(2./PI)*(T*T*ASIN(1./T)-SQRT(T*T-1.))
C COMPUTE THE INTEGRAL -I8
      CALL SG(AK,T,G,S,RG)
C COMPUTE THE INTEGRAL I12
      CALL SH(AK,T,G,S,RH)
      CALL SR1(AK,T,G,S,RR1)
      CALL SR2(AK,T,G,S,RR2)
C COMPUTE THE INTEGRAL -I15
       RI=T*T*R6-2.*RR1*SQRT(T*T-1.)/PI
C COMPUTE THE INTEGRAL -I16
      RJ=T*T*RH-2.*RR2*SQRT(T*T-1.)/PI
      ANUM1=R5*RH*RF-R5*R7*RJ
      ANUM2=R7*RG*RI-R6*RF*RG
      DEN=RI*RH-R6*RJ
C COMPUTE THE FIRST VARIATIONAL APPROXIMATION
      VAR1=R5*R7/R6
C COMPUTE THE SECOND VARIATIONAL APPROXIMATION
      VAR2=(ANUM1+ANUM2)/DEN
      WRITE(6,11)AK,T,G
   11 FORMAT(1X,*AK=*,F10.5,*T=*,F10.5,*G=*,F10.5)
      WRITE(6,12)VAR1,VAR2
   12 FORMAT(1X,*VAR1=*,2F10.5,*VAR2*,2F10.5)
      A1=CABS(VAR1)
      A2=CABS(VAR2)
      WRITE(6,147)A1,A2
  147 FORMAT(1X,*A1=*,F10.5,*A2=*,F10.5,/)
  310 CONTINUE
```

```
      END
      SUBROUTINE S5(AK,T,G,S,F)

      COMPLEX AI,S1,S2,S3,S8,HO,RES,F
      PI=3.1415926536
      AI=CMPLX(0.0,1.0)
      CALL SI1(AK,T,G,S1)
      CALL SI2(AK,T,G,S2)
      CALL SI3(AK,T,G,S3)
      X=AK*T*S
      X1=(1.0,0.0)*X
      CALL BESSI(0,X1,2,BJ,BY,HO,H2)
      A=SQRT(S*S-G*G)
      B=SQRT(S*S-1.)
      DEN=2*S*(2*S**2-1.)-2*S*A*B-S**3*A/B-S**3*B/A
      RES=PI*AI*HO*SIN (AK*S)*A/DEN
      S8=-S1-AI*S3+RES-AI*S2
      F=(1.-G*G)*S8/PI
      RETURN
      END
      SUBROUTINE SI1(AK,T,G,R)
      REAL I11,I12
      PI=3.1415926536
      M=50
      MM=M-1
      I11=0.0
      P=.02/EXP(1.)
      CALL HINT(AK,T,G,P,S1)
      R=.02*S1
      CALL HINT(AK,T,G,.02,S1)
      DO 105 I=1,MM,2
      Y2=.02+.98*I/M
      Y3=.02+.98*(I+1)/M
      CALL HINT(AK,T,G,Y2,S2)
      CALL HINT(AK,T,G,Y3,S3)
      I11=I11+(S1+4*S2+S3)
      S1=S3
  105 CONTINUE
      R=I11*.98/(3*M)+R
      IF ((50.-AK*T) .LT. 0.0) GO TO 123
      F=10./(AK*T)
      M=INT(F*10)
      MM=M-1
      H=(F-1.)/M
      DO 205 I=1,MM,2
      AM =M
      Y2=1.+I*H
      Y3=1.+(I+1)*H
```

```
      CALL HINT(AK,T,G,Y2,S2)
      CALL HINT(AK,T,G,Y3,S3)
      R=R+(S1+4*S2+S3)*H/3
      S1=S3
  205 CONTINUE
  123 R=R/PI
      RETURN
      END
      SUBROUTINE HINT(AK,T,G,TAU,BINT)
      COMPLEX S2,BJ,BY,H1,H2,AI
      AI=CMPLX(0.0,1.0)
      PI=3.1415926536
      V=AK*TAU
      A=SQRT(TAU*TAU+G*G)
      B=SQRT(TAU*TAU+1.)
      DEN=(TAU*TAU+.5)**2-TAU*TAU*A*B
      AEXP=EXP(V)-EXP(-V)
      S1=V*T
      S2=AI*S1
      CALL BESSI(0,S2,2,BJ,BY,H1,H2)
      BK=REAL(.5*PI*AI*H1)
      ANUM=BK*AEXP*A
      BINT=ANUM/DEN
  960 RETURN
      END
       SUBROUTINE SI2(AK,T,G,S2)
      COMPLEX S2,S21,S22,S23,H0
      COMPLEX X1,J1,Y,H2
      M=50
      MM=M-1
      S2=0.0
      S21=0.0
      DO 80 J=1,MM,2
      Y2=G+.05+(.90-G)*J/M
      Y3=G+.05+(.90-G)*(J+1)/M
      CALL CINT(AK,T,G,Y2,S22)
      CALL CINT(AK,T,G,Y3,S23)
      S2=S2+S21+4*S22+S23
      S21=S23
   80 CONTINUE
      S2=S2*(.90-G)/(3*M)
      N=50
      CALL CINT(AK,T,G,G,S21)
      DO 180 J=1,N
      Y2=G+.05*J/N
      CALL CINT(AK,T,G,Y2,S22)
      S2=S2+.025*(S21+S22)/N
      S21=S22
```

```
180 CONTINUE
    S21=0.0
    DO 195 J=1,N
    Y2=1.-.05*J/N
    CALL CINT(AK,T,G,Y2,S22)
    S2=S2+.025*(S21+S22)/N
    S21=S22
195  CONTINUE
    RETURN
    END
    SUBROUTINE CINT(AK,T,G,X,S2)
    COMPLEX ANUM2,S2,S1,RY,RJ,H0,H2
    S1=AK*T*X*(1.0,0.0)
    CALL BESSI(0,S1,2,RJ,RY,H0,H2)
    ANUM2=H0*SIN(AK*X)*X*X*SQRT(1.-X*X)*(X*X-G*G)
    DEN2=(X*X-.5)**4+X**4*(1.-X*X)*(X*X-G*G)
    S2=ANUM2/DEN2
    RETURN
    END
    SUBROUTINE SI3(AK,T,G,S3)
    COMPLEX S31,S32,S33,S3,R,S1
    M=50
    MM=M-1
    S=.02/EXP(1.)
    CALL DINT(AK,T,G,S,S1)
    R=.02*S1
    CALL DINT(AK,T,G,.02,S31)
    DO 100 J=1,MM,2
    Y2=(G-.07)*J/M+.02
    Y3=(G-.07)*(J+1.)/M+.02
    CALL DINT(AK,T,G,Y3,S33)
    CALL DINT(AK,T,G,Y2,S32)
    R=R+S31+4*S32+S33
    S31=S33
100 CONTINUE
    S3=R*(G-.07)/(3*M)
200 CONTINUE
    M=50
    X=G-.05
    CALL DINT(AK,T,G,X,S31)
    DO 220 I=1,49
    Y2=G-.05+.05*I/M
    CALL DINT(AK,T,G,Y2,S32)
    S3=S3+.025*(S31+S32)/M
    S31=S32
220 CONTINUE
300 CONTINUE
    RETURN
```

```
      END
      SUBROUTINE DINT(AK,T,G,X,S3)
      COMPLEX HO,ANUM,S3,Y1,BJ,BY,H2
      Y=AK*T*X
      Y1=(1.0,0.0)*Y
      CALL BESSI(0,Y1,2,BJ,BY,HO,H2)
      ANUM=HO*SIN(AK*X)*SQRT(G*G-X*X)
      DEN=(.5-X*X)**2+X*X*SQRT(1.-X*X)*SQRT(G*G-X*X)
      S3=ANUM/DEN
      RETURN
      END
      SUBROUTINE S6(AK,T,G,S,R)
      COMPLEX R,Y1,Y2,Y3,Z1,Z2,Z3
      PI=3.1415926536
      H=PI/100.
      R=0.0
      X1=0.0
      CALL F5(AK,G,X1,S,Z1)
      Y1=0.0
      DO 50 I=1,50,2
      X2=SIN(I*H)
      X3=SIN((I+1)*H)
      CALL F5(AK,G,X2,S,Z2)
      CALL F5(AK,G,X3,S,Z3)
      Y2=Z2*X2/(T*T-X2*X2)
      Y3=Z3*X3/(T*T-X3*X3)
      R=R+H*(Y1+4*Y2+Y3)/3
      Y1=Y3
   50 CONTINUE
      R=(2./PI)*SQRT(T*T-1.)*R
      RETURN
      END
      SUBROUTINE F5(AK,G,U,S,R)
      COMPLEX AI,R,Y23,Y33,SE,ANUM,RES,Y1,Y2,Y3
      PI=3.1415926536
      H=.001
      Y1=0.0
      R=0.0
      AI=CMPLX(0.0,1.0)
      DO 100 I=1,50
      X2=.001*I
      CALL AX5(G,X2,Y22)
      CALL ES5(AK,X2,U,Y23)
      Y2=Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
  100 CONTINUE
      H=(G-.1)/50
```

```
      DO 200 J=1,50,2
      X2=.05+J*H
      X3=.05+(J+1)*H
      CALL AX5(G,X2,Y22)
      CALL ES5(AK,X2,U,Y23)
      Y2=Y22*Y23
      CALL AX5(G,X3,Y32)
      CALL ES5(AK,X3,U,Y33)
      Y3=Y32*Y33
      R=R+(H/3.)*(Y1+4*Y2+Y3)
      Y1=Y3
  200 CONTINUE
      H=.001
      DO 300 I=1,100
      X2=G-.05+I*H
      CALL AX5(G,X2,Y22)
      CALL ES5(AK,X2,U,Y23)
      Y2=Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
  300 CONTINUE
      H=(.9-G)/50
      DO 400 I=1,50,2
      X2=G+.05+I*H
      X3=G+.05+(I+1)*H
      CALL AX5(G,X2,Y22)
      CALL ES5(AK,X2,U,Y23)
      Y2=Y22*Y23
      CALL AX5(G,X3,Y32)
      CALL ES5(AK,X3,U,Y33)
      Y3=Y32*Y33
      R=R+(H/3.)*(Y1+4*Y2+Y3)
      Y1=Y3
  400 CONTINUE
      H=.001
      DO 500 I=1,50
      X2=.95+I*H
      CALL AX5(G,X2,Y22)
      CALL ES5(AK,X2,U,Y23)
      Y2=Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
  500 CONTINUE
      CALL ES5(AK,S,U,SE)
      A=SQRT(S*S-G*G)
      B=SQRT(S*S-1.)
      ANUM=PI*A*SE
      ACEN=2.*(S*S-.5)*2.*S-2*S*A*B-S**3*A/B-S**3*B/A
```

```
      RES=ANUM/ADEN
      R=(1.-G*G)*AI*(R-RES)/(PI*PI)
      R=-R-1.
      RETURN
      END
      SUBROUTINE AX5(G,X,Y)
      IF (X .GE. G) GO TO 600
      A=SQRT(G*G-X*X)
      B=SQRT(1.-X*X)
      ANUM=A
      ADEN=(X*X-.5)**2+X*X*A*B
      Y=ANUM/ADEN
      GO TO 700
  600 ANUM =X**2*(X*X-G*G)*SQRT(1.-X*X)
      ADEN=(X*X-.5)**4+X**4*(X*X-G*G)*(1.-X*X)
      Y=ANUM/ADEN
  700 CONTINUE
      RETURN
      END
      SUBROUTINE ES5(AK,X,U,Y)
      COMPLEX UH1,J,BY,H1,H2
      COMPLEX E,Y,AI
      PI=3.1415926536
      AI=CMPLX(0.0,1.0)
      XH=AK*X
      E=CEXP(AI*XH)
      UH=U*XH
      UH1=(1.0,0.0)*UH
      CALL BESSI(0,UH1,1,J,BY,H1,H2)
      BJ=REAL(J)
      Y=AI*PI*(1.-E*BJ)
      RETURN
      END
      SUBROUTINE SG(AK,T,G,S,F)
      COMPLEX X1,BJ,BY,H2
      COMPLEX AI,S1,S2,S3,S5,HO,RES,F
      PI=3.1415926536
      AI=CMPLX(0.0,1.0)
      CALL SI4(AK,T,G,S1)
      CALL SI5(AK,T,G,S2)
      CALL SI6(AK,T,G,S3)
      X=AK*T*S
      X1=AI*X
      CALL BESSI(0,X1,2,BJ,BY,HO,H2)
      A=SQRT(S*S-G*G)
      B=SQRT(S*S-1.)
      DEN=2*S*(2*S**2-1.)-2*S*A*B-S**3*A/B-S**3*B/A
      SH=AK*S
```

```
      Q=(SH*COS(SH)+(SH*SH-1.)*SIN(SH))/(SH*SH)
      RES=PI*AI*HO*Q*A/DEN
      S5=-S1-AI*S3+RES-AI*S2
      F=(1.-G*G)*S5/PI
      RETURN
      END
      SUBROUTINE SI4(AK,T,G,R)
      REAL I11,I12
      PI=3.1415926536
      M=50
      MM=M-1
      I11=0.0
      S1=0.0
      DO 105 I=1,MM,2
      Y2=1.0*I/M
      Y3=1.0*(I+1)/M
      CALL HINT1(AK,T,G,Y2,S2)
      CALL HINT1(AK,T,G,Y3,S3)
      I11=I11+(S1+4*S2+S3)
      S1=S3
  105 CONTINUE
      R=I11/(3*M)
      IF ((50.-AK*T) .LT. 0.00) GO TO 127
      F=10./(AK*T)
      M=INT(F*10)
      MM=M-1
      H=(F-1.)/M
      DO 205 I=1,MM,2
      AM=M
      Y2=1.+I*H
      Y3=1.+(I+1)*H
      CALL HINT1(AK,T,G,Y2,S2)
      CALL HINT1(AK,T,G,Y3,S3)
      R=R+(S1+4*S2+S3)*H/3
      S1=S3
  205 CONTINUE
  127 R=2.*R/PI
      RETURN
      END
      SUBROUTINE HINT1(AK,T,G,TAU,BINT)
      COMPLEX S2,BJ,BY,HO,H2,AI
      AI=CMPLX(0.0,1.0)
      PI=3.1415926536
      V=AK*TAU
      A=SQRT(TAU*TAU+G*G)
      B=SQRT(TAU*TAU+1.)
      DEN=(TAU*TAU+.5)**2-TAU*TAU*A*B
      AEXP=(V*COSH(V)-(V*V+1.)*SINH(V))/(V*V)
```

```
      S1=V*T
      S2=AI*S1
      CALL BESSI(0,S2,2,BJ,BY,HO,H2)
      BK=REAL(.5*PI*AI*HO)
      ANUM=BK*AEXP*A
      BINT=ANUM/DEN
  960 RETURN
      END
      SUBROUTINE SI5(AK,T,G,S2)
      COMPLEX S2,S21,S22,S23
      M=50
      MM=M-1
      S2=0.0
      X=G+.05
      CALL CINT1(AK,T,G,X,S21)
      DO 80 J=1,MM,2
       Y2=G+.05+(.90-G)*J/M
      Y3=G+(.90-G)*(J+1)/M
      CALL CINT1(AK,T,G,Y2,S22)
      CALL CINT1(AK,T,G,Y3,S23)
      S2=S2+S21+4*S22+S23
      S21=S23
   80 CONTINUE
      S2=S2*(.90-G)/(3*M)
      N=50
      AI=CMPLX(0.0,1.0)
      CALL CINT1(AK,T,G,G,S21)
      DO 180 J=1,N
      Y2=G+.05*J/N
      CALL CINT1(AK,T,G,Y2,S22)
      S2=S2+.025*(S21+S22)/N
      S21=S22
  180 CONTINUE
      S21=0.0
      DO 195 J=1,N
      Y2=1.-.05*J/N
      CALL CINT1(AK,T,G,Y2,S22)
      S2=S2+.025*(S21+S22)/N
      S21=S22
  195 CONTINUE
      RETURN
      END
      SUBROUTINE CINT1(AK,T,G,X,S2)
      COMPLEX ANUM2,S2,S3,BJ,BY,HO,H2
      S1=AK*T*X
      S3=(1.0,0.0)*S1
      CALL BESSI(0,S3,2,BJ,BY,HO,H2)
      XH=AK*X
```

```
      Q=XH*COS(XH)+(XH*XH-1.)*SIN(XH)
      ANUM2=HO*Q*X*X*SQRT(1.-X*X)*(X*X-G*G)
      DEN2=((X*X-.5)**4+X**4*(1.-X*X)*(X*X-G*G))*XH*XH
      S2=ANUM2/DEN2
      RETURN
      END
      SUBROUTINE SI6(AK,T,G,S3)
      COMPLEX S31,S32,S33,S3,R
      M=50
      MM=M-1
      R=0.0
      S31=0.0
      DO 100 J=1,MM,2
      Y2=(G-.05)*J/M
      Y3=(G-.05)*(J+1.)/M
      CALL DINT1(AK,T,G,Y3,S33)
      CALL DINT1(AK,T,G,Y2,S32)
      R=R+4*S32+S33+S31
      S31=S33
  100 CONTINUE
      S3=R*(G-.07)/(3*M)
  200 CONTINUE
      M=50
      X=G-.05
      CALL DINT1(AK,T,G,X,S31)
      DO 220 I=1,49
      Y2=G-.05+.05*I/M
      CALL DINT1(AK,T,G,Y2,S32)
      S3=S3+.025*(S31+S32)/M
      S31=S32
  220 CONTINUE
  300 CONTINUE
       RETURN
       END
      SUBROUTINE DINT1(AK,T,G,X,S3)
      COMPLEX HO ,ANUM,S3,Y1,BJ,BY,H2
      Y=AK*T*X
      Y1=(1.0,0.0)*Y
      CALL BESSI(0,Y1,2,BJ,BY,HO,H2)
      XH=AK*X
      Q=XH*COS(XH)+(XH*XH-1.)*SIN(XH)
      ANUM =HO*Q*SQRT(G*G-X*X)
      DEN=((.5-X*X)**2+X*X*SQRT(1.-X*X)*SQRT(G*G-X*X))*XH*XH
      S3=ANUM/DEN
      RETURN
      END
      SUBROUTINE SH(AK,T,G,S,R)
      COMPLEX R,Y1,Y2,Y3,Z1,Z2,Z3
```

```
      PI=3.1415926536
      H=PI/100.
      R=0.0
      X1=0.0
      CALL F(AK,G,X1,S,Z1)
      Y1=0.0
      DO 50 I=1,50,2
      X2=SIN(I*H)
      X3=SIN((I+1)*H)
      CALL F(AK,G,X2,S,Z2)
      CALL F(AK,G,X3,S,Z3)
      Y2=Z2*X2/(T*T-X2*X2)
      Y3=Z3*X3/(T*T-X3*X3)
      R=R+H*(Y1+4*Y2+Y3)/3
      Y1=Y3
   50 CONTINUE
      R=2.*R*SQRT(T*T-1.)/PI
      RETURN
      END
      SUBROUTINE F(AK,G,U,S,R)
      COMPLEX AI,R,Y23,Y33,SE,ANUM,RES,Y1,Y2,Y3
      PI=3.1415926536
      H=.001
      Y1=0.0
      R=0.0
      AI=CMPLX(0.0,1.0)
      DO 100 I=1,50
      X2=.001*I
      CALL AX(G,X2,Y22)
      CALL ES(AK,X2,U,Y23)
      Y2=Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
  100 CONTINUE
      H=(G-.1)/50
      DO 200 J=1,50,2
      X2=.05+J*H
      X3=.05+(J+1.)*H
      CALL AX(G,X2,Y22)
      CALL ES(AK,X2,U,Y23)
      Y2=Y22*Y23
      CALL AX(G,X3,Y32)
      CALL ES(AK,X3,U,Y33)
      Y3=Y32*Y33
      R=R+(H/3.)*(Y1+4*Y2+Y3)
      Y1=Y3
  200 CONTINUE
      H=.001
```

```
      DO 300 I=1,100
      X2=G-.05+I*H
      CALL AX(G,X2,Y22)
      CALL ES(AK,X2,U,Y23)
      Y2=Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
  300 CONTINUE
      H=(.9-G)/50
      DO 400 I=1,50,2
      X2=G+.05+I*H
      X3=G+.05+(I+1)*H
      CALL AX(G,X2,Y22)
      CALL ES(AK,X2,U,Y23)
      Y2=Y22*Y23
      CALL AX(G,X3,Y32)
      CALL ES(AK,X3,U,Y33)
      Y3=Y32*Y33
      R=R+(H/3.)*(Y1+4*Y2+Y3)
      Y1=Y3
  400 CONTINUE
      H=.001
      DO 500 I=1,50
      X2=.95+I*H
      CALL AX(G,X2,Y22)
      CALL ES(AK,X2,U,Y23)
      Y2=Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
  500 CONTINUE
      CALL ES(AK,S,U,SE)
      A=SQRT(S*S-G*G)
      B=SQRT(S*S-1.)
      ANUM=PI*S*A*SE
      ADEN=2.*(S*S-.5)*2.*S-2*S*A*B-S**3*A/B-S**3*B/A
      RES=ANUM/ADEN
      R=(1.-G*G)*.5*AI*AK*(R-RES)/PI
      R=(-R-(1.+.5*U*U))/2.
      RETURN
      END
      SUBROUTINE AX(G,X,Y)
      IF (X .GE. G)GO TO 600
      A=SQRT(G*G-X*X)
      B=SQRT(1.-X*X)
      ANUM=X*A
      ADEN=(X*X-.5)**2+X*X*A*B
      Y=ANUM/ADEN
      GO TO 700
```

```
600 ANUM=X**3*(X*X-G*G)*SQRT(1.- X*X)
    ADEN=(X*X-.5)**4+X**4*(X*X-G*G)*(1.-X*X)
    Y=ANUM/ADEN
700 CONTINUE
    RETURN
    END
    SUBROUTINE ES(AK,X,U,Y)
    COMPLEX UH1,J,BY,H1,H2
    COMPLEX AI,Y,A1,A2,A3,A4,Y1
    PI=3.1415926536
    AI=CMPLX(0.0,1.0)
    XH=AK*X
    IF (XH .LE. .0004) GO TO 17
    A1=-2./(AI*XH)-4.*AI/(XH**3)-U*U/(AI*XH)
    UH=XH*U
    UH1=(1.0,0.0)*UH
    CALL BESSI(0,UH1,1,J,BY,H1,H2)
    BJ=REAL(J)
    A2=CEXP(AI*XH)
    A3=2./(AI*XH)+2./(XH*XH)+2.*AI/(XH**3)
    A4=2.*A2*A3*BJ
    Y=A1+A4
    GO TO 18
 17 CONTINUE
    C1=8./3.
    C2=-.5+U*U/2.+(U**4)/16.
    C3=-8./15.-2*U*U/3.
    Y=C1+AI*C2*XH+C3*XH**3
 18 CONTINUE
    RETURN
    END
    SUBROUTINE SR1(AK,U,G,S,R)
    COMPLEX AI,R,Y23,Y33,SE,ANUM,RES,Y1,Y2,Y3
    PI=3.1415926536
    H=.001
    Y1=0.0
    R=0.0
    AI=CMPLX(0.0,1.0)
    DO 100 I=1,50
    X2=.001*I
    CALL AX2(G,X2,Y22)
    CALL ES2(AK,X2,U,Y23)
    Y2=Y22*Y23
    R=R+.5*H*(Y1+Y2)
    Y1=Y2
100 CONTINUE
    H=(G-.1)/50
    DO 200 J=1,50,2
```

```
      X2=.05+J*H
      X3=.05+(J+1.)*H
      CALL AX2(G,X2,Y22)
      CALL ES2(AK,X2,U,Y23)
      Y2=Y22*Y23
      CALL AX2(G,X3,Y32)
      CALL ES2(AK,X3,U,Y33)
      Y3=Y32*Y33
      R=R+(H/3.)*(Y1+4*Y2+Y3)
      Y1=Y3
  200 CONTINUE
      H=.001
      DO 300 I=1,100
      X2=G-.05+I*H
      CALL AX2(G,X2,Y22)
      CALL ES2(AK,X2,U,Y23)
      Y2=Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
  300 CONTINUE
      H=(.9-G)/50
      DO 400 I=1,50,2
      X2=G+.05+I*H
      X3=G+.05+(I+1)*H
      CALL AX2(G,X2,Y22)
      CALL ES2(AK,X2,U,Y23)
      Y2=Y22*Y23
      CALL AX2(G,X3,Y32)
      CALL ES2(AK,X3,U,Y33)
      Y3=Y32*Y33
      R=R+(H/3.)*(Y1+4*Y2+Y3)
      Y1=Y3
  400 CONTINUE
      H=.001
      DO 500 I=1,50
      X2=.95+I*H
      CALL AX2(G,X2,Y22)
      CALL ES2(AK,X2,U,Y23)
      Y2=Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
  500 CONTINUE
      CALL ES2(AK,S,U,SE)
      A=SQRT(S*S-G*G)
      B=SQRT(S*S-1.)
      ANUM=PI*A*SE
      ADEN=2.*(S*S-.5)*2.*S-2*S*A*B-S**3*A/B-S**3*B/A
      RES=ANUM/ADEN
```

```
      R=-1.*(1.-G*G)*.5*AI*(R-RES)/PI
      RETURN
      END
      SUBROUTINE AX2(G,X,Y)
      IF (X .GE. G) GO TO 600
      A=SQRT(G*G-X*X)
      B=SQRT(1.-X*X)
      ANUM=A
      ADEN=(X*X-.5)**2+X*X*A*B
      Y=ANUM/ADEN
      GO TO 700
  600 ANUM=X**2*(X*X-G*G)*SQRT(1.-X*X)
      ADEN=(X*X-.5)**4+X**4*(X*X-G*G)*(1.-X*X)
      Y=ANUM/ADEN
  700 CONTINUE
      RETURN
      END
      SUBROUTINE ES2(AK,X,U,Y)
      COMPLEX AI,E,Y
      AI=CMPLX(0.0,1.0)
      XH=AK*X
      E=CEXP(2.*AI*XH)
      Y=(1.-E)/XH
      RETURN
      END
      SUBROUTINE SR2(AK,U,G,S,R)
      COMPLEX R,Y1,Y2,Y3,Z1,Z2,Z3
      PI=3.1415926536
      H=PI/100.
      R=0.0
      X1=0.0
      CALL F(AK,G,X1,S,Z1)
      Y1=0.0
      DO 52 I=1,50,2
      X2=SIN(I*H)
      X3=SIN((I+1)*H)
      CALL F(AK,G,X2,S,Z2)
      CALL F(AK,G,X3,S,Z3)
      Y2=Z2*X2
      Y3=Z3*X3
      R=R+H*(Y1+4*Y2+Y3)/3.
      Y1=Y3
   52 CONTINUE
      R=-1.*R
      RETURN
      END
      SUBROUTINE BESSI(ORDER,Z,M,J,Y,H1,H2)
      INTEGER ORDER
```

```
      COMPLEX H1,H2,J,P,Q,T,TO,Z,Z1,A,Y
      DIMENSION AJ(2,8),AP(2,6),AQ(2,6),AY(2,9)
      DATA AJ/1.0,1.9999999998,-3.9999998721,-3.9999999710,
     13.9999973021,2.6666660544,-1.7777560599,-.8888839649,
     2.4443584263,.1777582922,-.0709253492,-.0236616773,
     3.0076771853,.0022069155,-.0005014415,-.0001289769/,
     4AP/.3989422793,.3989422819,-.0017530620,.0029218256,
     5.0001734300,-.0002232030,-.0000487613,.0000580759,
     6.0000173565,-.0000200920,-.0000037043,.0000042414/,
     7AQ/-.0124669441,.0374008364,.0004564324,-.0006390400,
     8-.0000869791,.0001064741,.0000342468,-.0000398708,
     9-.0000200920,.0000162200,.0000032312,-.0000036594/
      DATA AY/-.5772156649,1.0000000004,-1.6911374142,
     1-.6177253972,3.691138879,-10.7645472724,-2.2331100234,
     13.691138879,-10.7645472724,-2.2331102234,
     211.6207891416,.6694321484,-4.9105291148,-.1214187561,
     3.6694321484,-4.9105291148,-.1214187561,
     31.1418033012,.0148999271,-1691081720,-.0013508487,
     4.0148999271,-.1691081720,-.0013508487,.0169921876,
     4 .0169921876,.0000891322,.00102663368/
      PI=3.14159265359
      T=Z/4
      IF(CABS(T) .GT. 1.0) GO TO 130
      J=(0.0,0.0)
      TO=(1.0,0.0)
      DO 100 I=1,8
      J=J+AJ(ORDER+1,I)*TO
  100 TO=TO*T*T
      IF(ORDER .EQ. 1) J=T*J
      GO TO (125,200,200),M
  125 RETURN
  130 T=1/T
      P=(0.0,0.0)
      Q=(0.0,0.0)
      TO=(1.0,0.0)
      DO 140 I=1,6
      P=P+AP(ORDER+1,I)*TO
      Q=Q+AQ(ORDER+1,I)*TO
  140 TO=TO*T*T
      P=SQRT(2*PI)*P
      Q=Q*T*SQRT(2*PI)
      N=(REAL(Z)-PI/4-ORDER*PI/2)/(2*PI)
      A=CSQRT(2/(PI*Z))
      Z1=Z-PI/4-ORDER*PI/2-2*PI*N
      J=A*(P*CCOS(Z1)-Q*CSIN(Z1))
      Y=A*(P*CSIN(Z1)+Q*CCOS(Z1))
      H1=J+(0.0,1.0)*Y
      H2=J-(0.0,1.0)*Y
```

```
      RETURN
200   Y=(0.0,0.0)
      TO=(1.0,0.0)
      DO 210 I=1,9
      Y=Y+AY(ORDER+1,I)*TO
210   TO=TO*T*T
      IF (ORDER .EQ.1) Y=Y/Z
      Y=2/PI*(J*CLOG(T*2)-Y)
      H1=J+(0.0,1.0)*Y
      H2=J-(0.0,1.0)*Y
      END
```

```
      PROGRAM UR(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      COMPLEX R1,R2,RB,RC,RE,ANUM1,ANUM2,ADEN2,UR1,UR2
C THIS PROGRAM COMPUTES THE FIRST AND SECOND VARIATIONAL
C APPROXIMATIONS FOR UR
      READ *,N,AK
      G=.57735
      S=1.087663874
      WRITE(6,29)
29    FORMAT( * APPROXIMATION FOR UR*)
      DO 69 K=1,N
      READ *,T
      PI=3.1415926536
C COMPUTE THE INTEGRAL I5
      CALL U1(AK,T,G,S,R1)
C COMPUTE THE INTEGRAL I9
      CALL U2(AK,T,G,S,R2)
C COMPUTE THE INTEGRAL I1
      R4=-2.*G*G/(PI*T)
C COMPUTE THE INTEGRAL I2
      RA=-4.*G*G/(3.*PI*T)
C COMPUTE THE INTEGRAL I6
      CALL UB(AK,T,G,S,RB)
C COMPUTE THE INTEGRAL I13=I14
      CALL UC(AK,T,G,S,RC)
C COMPUTE THE INTEGRAL I10
      CALL UE(AK,T,G,S,RE)
C COMPUTE THE SECOND VARIATIONAL APPROXIMATION
      ANUM1=R1*RC*RA-R1*R4*RE
      ANUM2=RB*R4*RC-RB*R2*RA
      ADEN2=RC*RC-R2*RE
      UR2=(ANUM1+ANUM2)/ADEN2
C COMPUTE THE FIRST VARIATIONAL APPROXIMATION
      UR1=R1*R4/R2
      WRITE(6,100)AK,T,G
100   FORMAT(1X,*AK=*,F10.5,*T=*,F10.5,*G=*,F10.5)
      WRITE(6,200)UR1,UR2
200   FORMAT(1X,*1ST VAR APP=*,2F10.5,*2ND VAR APP=*,2F10.5)
      A1=CABS(UR1)
      A2=CABS(UR2)
      WRITE(6,205)A1,A2
205   FORMAT(1X,*A1=*,F10.5,*A2=*,F10.5,/)
63    CONTINUE
69    CONTINUE
      END
      SUBROUTINE U1(AK,T,G,S,F)
      COMPLEX AI,HS1,RES,Y1,Y2,Y3,ANUM,RES1,F
      COMPLEX S2,J1,Y,H2
```

```
      AI=CMPLX(0.0,1.0)
      PI=3.1415926536
      RES=0.0
      X1=G+.05*(1.-G)/50
      CALL AIN1(AK,T,G,S,X1,Y1)
      DO 101 I=1,50
      X2=G+(I)*.05*(1.-G)/50
      CALL AIN1(AK,T,G,S,X2,Y2)
      RES=RES+(.5*.05*(1.-G)/50)*(Y1+Y2)
      Y1=Y2
101   CONTINUE
      DO 104 J=1,100,2
      X2=G+.05*(1.-G)+J*(.95-G)/100
      X3=G+.05*(1.-G)+(J+1)*(.95-G)/100
      CALL AIN1(AK,T,G,S,X2,Y2)
      CALL AIN1(AK,T,G,S,X3,Y3)
      RES=RES+((.95-G)/100/3)*(Y1+4*Y2+Y3)
      Y1=Y3
104   CONTINUE
      A=SQRT(S*S-G*G)
      B=SQRT(S*S-1)
      AEXP=SIN(AK*S)
      S1=AK*S*T
      S2=(1.0,0.0)*S1
      CALL BESSI(1,S2,2,J1,Y,HS1,H2)
      ANUM=(2*S*S-1-2*A*B)*HS1*AEXP*S
      ADEN=(2*(S*S-.5)*2*S-2*S*A*B-S**3*A/B-S**3*B/A)
      RES1=ANUM/ADEN
      F=- RES+PI* RES1
      F=F*(1.-G*G)*AI/(PI)
      RETURN
      END
      SUBROUTINE AIN1(AK,T,G,S,X,Y)
      COMPLEX H,ANUM,Y,X2,Y1,J1,H2
      AEXP=SIN(AK*X)
      X1=AK*T*X
      X2=(1.0,0.0)*X1
      CALL BESSI(1,X2,2,J1,Y1,H,H2)
      A=SQRT(X*X-G*G)
      B=SQRT(1.-X*X)
      ANUM=(X*X-.5)*A*B*H*AEXP*X
      ADEN=((X*X-.5)**4+X**4*(X*X-G*G)*(1.-X*X))
      Y=ANUM/ADEN
      RETURN
      END
      SUBROUTINE U2(AK,U,G,S,R)
      COMPLEX AI,R,Y23,Y33,SE,ANUM,RES,Y1,Y2,Y3
```

```
      PI=3.1415926536
      H=.001
      Y1=0.0
      R=0.0
      AI=CMPLX(0.0,1.0)
      DO 100 I=1,50
      X2=.001*I
      CALL AX(G,X2,Y22)
      CALL ES(AK,X2,U,Y23)
      Y2= Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
100   CONTINUE
      H=(G-.1)/100
      DO 200 J=1,100,2
      X2=.05+J*H
      X3=.05+(J+1.)*H
      CALL AX(G,X2,Y22)
      CALL ES(AK,X2,U,Y23)
      Y2= Y22*Y23
      CALL AX(G,X3,Y32)
      CALL ES(AK,X3,U,Y33)
      Y3=  Y32*Y33
      R=R+(H/3.)*(Y1+4*Y2+Y3)
      Y1=Y3
200   CONTINUE
      H=.001
      DO 300 I=1,100
      X2=G-.05+I*H
      CALL AX(G,X2,Y22)
      CALL ES(AK,X2,U,Y23)
      Y2= Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
300   CONTINUE
      H=(.9-G)/100
      DO 400 I=1,100,2
      X2=G+.05+I*H
      X3=G+.05+(I+1)*H
      CALL AX(G,X2,Y22)
      CALL ES(AK,X2,U,Y23)
      Y2= Y22*Y23
      CALL AX(G,X3,Y32)
      CALL ES(AK,X3,U,Y33)
      Y3= Y32*Y33
      R=R+(H/3.)*(Y1+4*Y2+Y3)
      Y1=Y3
```

```
400   CONTINUE
      H=.001
      DO 500 I=1,50
      X2=.95+I*H
      CALL AX(G,X2,Y22)
      CALL ES(AK,X2,U,Y23)
      Y2= Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
500   CONTINUE
      CALL ES(AK,S,U,SE)
      A=SQRT(S*S-G*G)
      B=SQRT(S*S-1.)
      ANUM=PI* A* SE
      ADEN=2.*(S*S-.5)*2.*S-2*S*A*B-S** 3*A/B-S**3*B/A
      RES=ANUM/ADEN
      R=(1.-G*G)*.5*AI*  (R-RES)/PI
      R=-2.*G*G*R/(U*PI)
      RETURN
      END
      SUBROUTINE AX(G,X,Y)
      IF(X .GE. G)GO TO 600
      A=SQRT(G*G-X*X)
      B=SQRT(1.-X*X)
      ANUM=A
      ADEN=(X*X-.5)**2+X*X*A*B
      Y=ANUM/ADEN
      GOTO 700
600   ANUM=X**2*(X*X-G*G)*SQRT(1.-X*X)
      ADEN=(X*X-.5)**4+X**4*(X*X-G*G)*(1.-X*X)
      Y=ANUM/ADEN
700   CONTINUE
      RETURN
      END
      SUBROUTINE ES(AK,X,U,Y)
      COMPLEX AI,E,Y
      AI=CMPLX(0.0,1.0)
      XH=AK*X
      E=CEXP(2.*AI*XH)
      Y=(1.-E)/XH
      RETURN
      END
      SUBROUTINE UB(AK,T,G,S,F)
      COMPLEX AI,HS1,RES,Y1,Y2,Y3,ANUM,RES1,F
      COMPLEX S2,J1,Y,H2
      AI=CMPLX(0.0,1.0)
      PI=3.1415926536
```

```
      RES=0.0
      X1=G+.05*(1.-G)/50
      CALL AIN2(AK,T,G,S,X1,Y1)
      DO 101 I=1,50
      X2=G+(I)*.05*(1.-G)/50
      CALL AIN2(AK,T,G,S,X2,Y2)
      RES=RES+(.5*.05*(1.-G)/50)*(Y1+Y2)
      Y1=Y2
101   CONTINUE
      DO 104 J=1,100,2
      X2=G+.05*(1.-G)+J*(.95-G)/100
      X3=G+.05*(1.-G)+(J+1)*(.95-G)/100
      CALL AIN2(AK,T,G,S,X2,Y2)
      CALL AIN2(AK,T,G,S,X3,Y3)
      RES=RES+((.95-G)/100/3)*(Y1+4*Y2+Y3)
      Y1=Y3
104   CONTINUE
      A=SQRT(S*S-G*G)
      B=SQRT(S*S-1)
      AEXP=AK*S*COS(AK*S)+(AK*AK*S*S-1)*SIN(AK*S)
      S1=AK*S*T
      S2=(1.0,0.0)*S1
      CALL BESSI(1,S2,2,J1,Y,HS1,H2)
      ANUM=(2*S*S-1-2*A*B)*HS1*AEXP
      ADEN=(2*(S*S-.5)*2*S-2*S*A*B-S**3*A/B-S**3*B/A)*S
      RES1=ANUM/ADEN
      F=-AI*RES+PI*AI*RES1
      F= (1.-G*G)*F/(PI*AK*AK)
      RETURN
      END
      SUBROUTINE AIN2(AK,T,G,S,X,Y)
      COMPLEX X2,Y1,J1,H2
      COMPLEX H,ANUM,Y
      AEXP=AK*X*COS(AK*X)+(AK*AK*X*X-1)*SIN(AK*X)
      X1=AK*T*X
      X2=(1.0,0.0)*X1
      CALL BESSI(1,X2,2,J1,Y1,H,H2)
      A=SQRT(X*X-G*G)
      B=SQRT(1.-X*X)
      ANUM=(X*X-.5)*A*B*H*AEXP
      ADEN=((X*X-.5)**4+X**4*(X*X-G*G)*(1.-X*X))*X
      Y=ANUM/ADEN
      RETURN
      END
      SUBROUTINE F1(AK,G,U,S,R)
      COMPLEX AI,R,Y23,Y33,SE,ANUM,RES,Y1,Y2,Y3
      PI=3.1415926536
```

```
      H=.001
      Y1=0.0
      R=0.0
      AI=CMPLX(0.0,1.0)
      DO 100 I=1,50
      X2=.001*I
      CALL AX1(G,X2,Y22)
      CALL ES1(AK,X2,U,Y23)
      Y2= Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
100   CONTINUE
      H=(G-.1)/100
      DO 200 J=1,100,2
      X2=.05+J*H
      X3=.05+(J+1.)*H
      CALL AX1(G,X2,Y22)
      CALL ES1(AK,X2,U,Y23)
      Y2= Y22*Y23
      CALL AX1(G,X3,Y32)
      CALL ES1(AK,X3,U,Y33)
      Y3=  Y32*Y33
      R=R+(H/3.)*(Y1+4*Y2+Y3)
      Y1=Y3
200   CONTINUE
      H=.001
      DO 300 I=1,100
      X2=G-.05+I*H
      CALL AX1(G,X2,Y22)
      CALL ES1(AK,X2,U,Y23)
      Y2= Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
300   CONTINUE
      H=(.9-G)/100
      DO 400 I=1,100,2
      X2=G+.05+I*H
      X3=G+.05+(I+1)*H
      CALL AX1(G,X2,Y22)
      CALL ES1(AK,X2,U,Y23)
      Y2= Y22*Y23
      CALL AX1(G,X3,Y32)
      CALL ES1(AK,X3,U,Y33)
      Y3= Y32*Y33
      R=R+(H/3.)*(Y1+4*Y2+Y3)
      Y1=Y3
400   CONTINUE
```

```
      H=.001
      DO 500 I=1,50
      X2=.95+I*H
      CALL AX1(G,X2,Y22)
      CALL ES1(AK,X2,U,Y23)
      Y2= Y22*Y23
      R=R+.5*H*(Y1+Y2)
      Y1=Y2
500   CONTINUE
      CALL ES1(AK,S,U,SE)
      A=SQRT(S*S-G*G)
      B=SQRT(S*S-1.)
      ANUM=PI*S*A* SE
      ADEN=2.*(S*S-.5)*2.*S-2*S*A*B-S**3*A/B-S**3*B/A
      RES=ANUM/ADEN
      R=(1.-G*G)*.5*AI*AK*(R-RES)/PI
      R=(-R-(1.+.5*U*U))/(2.)
      RETURN
      END
      SUBROUTINE AX1(G,X,Y)
      IF(X .GE. G)GO TO 600
      A=SQRT(G*G-X*X)
      B=SQRT(1.-X*X)
      ANUM=X*A
      ADEN=(X*X-.5)**2+X*X*A*B
      Y=ANUM/ADEN
      GOTO 700
600   ANUM=X**3*(X*X-G*G)*SQPT(1.-X*X)
      ADEN=(X*X-.5)**4+X**4*(X*X-G*G)*(1.-X*X)
      Y=ANUM/ADEN
700   CONTINUE
      RETURN
      END
      SUBROUTINE ES1(AK,X,U,Y)
      COMPLEX UH1,J,Y5,H1,H2
      COMPLEX AI,Y,A1,A2,A3,A4,Y1
      PI=3.1415926536
      AI=CMPLX(0.0,1.0)
      XH=AK*X
      IF (XH .LE. .0004) GO TO 17
      A1=-2./(AI*XH)-4.*AI/(XH**3)-U*U/(AI*XH)
      UH=XH*U
      UH1=UH*(1.0,0.0)
      CALL BESSI(0,UH1,1,J,Y5,H1,H2)
      BJ=REAL(J)
      BJ=REAL(J)
      A2=CEXP(AI*XH)
```

```
      A3=2./(AI*XH)+2./(XH*XH)+2.*AI/(XH**3)
      A4=2.*A2*A3*BJ
      Y=A1+A4
      GO TO 18
17    CONTINUE
      C1=8./3.
      C2=-.5+U*U/2.+(U**4)/16.
      C3=-8./15.-2*U*U/3.
      Y=C1+AI*C2*XH+C3*XH**3
18    CONTINUE
      RETURN
      END
      SUBROUTINE UC(AK,U,G,S,R)
      COMPLEX R,Y1,Y2,Y3,Z1,Z2,Z3
      PI=3.1415926536
      H=PI/100.
      R=0.0
      X1=0.0
      CALL F1(AK,G,X1,S,Z1)
      Y1=0.0
      DO 52 I=1,50,2
      X2=SIN(I*H)
      X3=SIN((I+1)*H)
      CALL F1(AK,G,X2,S,Z2)
      CALL F1(AK,G,X3,S,Z3)
      Y2=Z2*X2
      Y3=Z3*X3
      R=R+H*(Y1+4*Y2+Y3)/3.
      Y1=Y3
52    CONTINUE
      R=2.*G*G*R/(PI*U)
      RETURN
      END
      SUBROUTINE UE(AK,T,G,S,R)
      COMPLEX AI,Y22,Y32,Y1,Y2,Y3,A,ANUM,RES,R
      AI=CMPLX(0.0,1.0)
      PI=3.1415926536
      R=0.0
      H=.05/50
      Y1=AK*G
      DO 100 I=1,50
      X1=I*.05/50
      CALL AM(G,X1,Y21)
      CALL AG(AK,G,X1,Y22)
      Y2=Y21*Y22
      R=R+H*(Y1+Y2)/2
      Y1=Y2
```

```
100   CONTINUE
      H=(G-.1)/100
      DO 200 I=1,100,2
      X1=.05+I*H
      X2=.05+(I+1)*H
      CALL AM(G,X1,Y21)
      CALL AG(AK,G,X1,Y22)
      Y2=Y21*Y22
      CALL AM(G,X2,Y31)
      CALL AG(AK,G,X2,Y32)
      Y3=Y31*Y32
      R=R+H*(Y1+4*Y2+Y3)/3
      Y1=Y3
200   CONTINUE
      H=.05/50
      DO 300 I=1,50
      X1=(G-.05)+I*H
      CALL AM(G,X1,Y21)
      CALL AG(AK,G,X1,Y22)
      Y2=Y21*Y22
      R=R+H*(Y1+Y2)/2
      Y1=Y2
300   CONTINUE
      DO 400 I=1,50
      X1=G+I*H
      CALL AM(G,X1,Y21)
      CALL AG(AK,G,X1,Y22)
      Y2=Y21*Y22
      R=R+H*(Y1+Y2)/2
      Y1=Y2
400   CONTINUE
      H=(.90-G)/100
      DO 500 I=1,100,2
      X1=G+.05+I*H
      X2=G+.05+(I+1)*H
      CALL AM(G,X1,Y21)
      CALL AG(AK,G,X1,Y22)
      Y2=Y21*Y22
      CALL AM(G,X2,Y31)
      CALL AG(AK,G,X2,Y32)
      Y3=Y31*Y32
      R=R+H*(Y1+4*Y2+Y3)/3
      Y1=Y3
500   CONTINUE
      H=.05/50
      DO 600 I=1,50
      X1=.95+I*H
```

```
      CALL AM(G,X1,Y21)
      CALL AG(AK,G,X1,Y22)
      Y2=Y21*Y22
      R=R+(Y1+Y2)*H/2
      Y1=Y2
600   CONTINUE
      CALL AG(AK,G,S,A)
      ANUM=PI*SQRT(S*S-G*G)*A
      A1=SQRT(S*S-G*G)
      A2=SQRT(S*S-1)
      ADEN=2*S*(2*S*S-1.)-2*S*A1*A2-S**3*A1/A2-S**3*A2/A1
      RES=ANUM/ADEN
      R=R-RES-2.*PI/(15.*(1.-G*G))
      R=-G*G*(1.-G*G)*R/(PI*PI*PI*T)
      END
      SUBROUTINE AM(G,X,Y)
      IF ((X-G) .LT. 0.) GO TO 800
      ANUM=X*X*(X*X-G*G)*SQRT(1.-X*X)
      ADEN=(X*X-.5)**4+X**4*(X*X-G*G)*(1.-X*X)
      Y=ANUM/ADEN
      GO TO 850
800   ANUM=SQRT(G*G-X*X)
      ADEN=(X*X-.5)**2+X*X*SQRT(G*G-X*X)*SQRT(1.-X*X)
      Y=ANUM/ADEN
850   CONTINUE
      RETURN
      END
      SUBROUTINE AG(AK,G,X,Y)
      COMPLEX Y,AI,X2,T1,S1,S2,R
      AI=CMPLX(0.0,1.0)
      X1=AK*X
      X2=X1*AI
      T1=-.5*AI*(1.-CEXP(2.*X2))/X1
      S1=2*CEXP(X2)
      CALL B(AK,X1,R)
      C=SIN(X1)/(X1**3)-COS(X1)/(X1**2)
      Y=T1+S1*C-AI*X1*R/8.
      RETURN
      END
      SUBROUTINE B(AK,X1,R)
      COMPLEX AI,R,E,T1,T2,S1,S2,U1,U2
      AI=CMPLX(0.0,1.0)
      IF (X1 .LE. .2393333)GO TO 900
      E=CEXP( AI*X1)
      T1=2.*(8.*AI/(15.*X1)-.5/(X1*X1)+4*AI/(3*(X1**3)))
      T2=(-2/(X1*X1)-2*AI/(X1**3))*E
      S1=T2+AI/X1+2*AI/(X1**3)
```

```
      S2=T2-AI/X1-2.*AI/(X1**3)
      R=T1+S1*S2
      GO TO 950
900   U1=.8888888-.4571424*AI*X1+.1777604*X1*X1
      U2=.0564372*AI*X1*X1*X1
      R=U1+U2
950   RETURN
      END
      SUBROUTINE BESSI(ORDER,Z,M,J,Y,H1,H2)
      INTEGER ORDER
      COMPLEX H1,H2,J,P,Q,T,TO,Z,Z1,A,Y
      DIMENSION AJ(2,8),AP(2,6),AQ(2,6),AY(2,9)
      DATA AJ/1.0,1.9999999998,-3.9999998721,-3.9999999710,
     +3.9999973021,2.6666660544,-1.7777560599,-.8888839649,
     +.4443584263,.1777582922,-.0709253492,-.0236616773,
     +.0076771853,.0022069155,-.0005014415,-.0001289769/,
     +AP/.3989422793,.3989422819,-.0017530620,.0029218256,
     +.0001734300,-.0002232030,-.0000487613,.0000580759,
     +.0000173565,-.0000200920,-.0000037043,.0000042414/,
     +AQ/-.0124669441,.0374008364,.0004564324,-.0006390400,
     +-.0000869791,.0001064741,.0000342468,-.0000398708,
     +-.0000200920,.0000162200,.0000032312,-.0000036594/
      DATA AY/-.5772156649,1.0000000004,
     +-1.6911374142,-.6177253972,3.691138879,
     +-10.7645472724,-2.2331102234,11.6207891416
     + ,.6694321484,-4.9105291148,-.1214187561,1.1418033012,
     + .0148999271,-.1691081720,-.0013508487,.0169921876.
     + .0000891322,.0010266336 8/
      PI=3.14159265359
      T=Z/4
      IF(CABS(T).GT.1.0)GO TO 130
      J=(0.0,0.0)
      TO=(1.0,0.0)
      DO 100 I=1,8
      J=J+AJ(ORDER+1,I)*TO
100   TO=TO*T*T
      IF(ORDER .EQ.1) J=T*J
      GO TO (125,200,200),M
125   RETURN
130   T=1/T
      P=(0.0,0.0)
      Q=(0.0,0.0)
      TO=(1.0,0.0)
      DO 140 I=1,6
      P=P+AP(ORDER+1,I)*TO
      Q=Q+AQ(ORDER+1,I)*TO
140   TO=TO*T*T
```

```
      P=SQRT(2*PI)*P
      Q=Q*T*SQRT(2*PI)
      N=(REAL(Z)-PI/4-ORDER*PI/2)/(2*PI)
      A=CSQRT(2/(PI*Z))
      Z1=Z-PI/4-ORDER*PI/2-2*PI*N
      J=A*(P*CCOS(Z1)-Q*CSIN(Z1))
      Y=A*(P*CSIN(Z1)+Q*CCOS(Z1))
      H1=J+(0.0,1.0)*Y
      H2=J-(0.0,1.0)*Y
      RETURN
200   Y=(0.0,0.0)
      TO=(1.0,0.0)
      DO 210 I=1,9
      Y=Y+AY(ORDER+1,I)*TO
210   TO=TO*T*T
      IF(ORDER.EQ.1)Y=Y/Z
      Y=2/PI*(J*CLOG(T*2)-Y)
      H1=J+(0.0,1.0)*Y
      H2=J-(0.0,1.0)*Y
      RETURN
      END
```

APPENDIX E

TABLES

Table 1. First and Second Variational Approximations for U_r. ($ak = .2, .5, \gamma^2 = 1/3$)

ak = .2 / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_r}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_r}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_r}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_r}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_r}{d}\right\rvert$
1.25	-.1786	-.1786	.0181	.0181	.1796
2.0	-.1199	-.1199	.0028	.0028	.1200
5.0	-.0557	-.0557	-.0269	-.0269	.0618
10.0	-.0056	-.0056	-.0418	-.0418	.0422
20.0	.0292	.0292	.0114	.0114	.0314
30.0	-.0199	-.0199	.0172	.0172	.0263
40.0	-.0048	-.0048	-.0211	-.0211	.0216
50.0	.0161	.0161	.0053	.0053	.0169

ak = .5 / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_r}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_r}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_r}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_r}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_r}{d}\right\rvert$
1.25	-.2104	-.2104	.0216	.0216	.2115
2.0	-.1504	-.1504	-.0332	-.0332	.1540
5.0	.0143	.0143	-.0938	-.0938	.0949
10.0	.0280	.0280	.0660	.0660	.0717
20.0	.0421	.0421	.0034	.0034	.0422
30.0	.0272	.0272	-.0128	-.0128	.0301
40.0	.0068	.0068	-.0328	-.0328	.0335
50.0	-.0167	-.0167	-.0193	-.0193	.0255

Table 2. First and Second Variational Approximations for U_r. ($ak = 1.0,\ 1.5,\ \gamma^2 = 1/3$)

$\alpha = 1.0$ / t	$\mathrm{Re}\left(\frac{U_r}{d}\right)^1$	$\mathrm{Re}\left(\frac{U_r}{d}\right)^2$	$\mathrm{Im}\left(\frac{U_r}{d}\right)^1$	$\mathrm{Im}\left(\frac{U_r}{d}\right)^2$	$\left\lvert\frac{U_r}{d}\right\rvert^2$
1.25	-.2642	-.2643	-.0222	-.0222	.2652
2.0	-.1387	-.1388	-.1545	-.1545	.2076
5.0	.1001	.1002	.1030	.1030	.1437
10.0	.0795	.0795	-.0258	-.0258	.0836
20.0	-.0113	-.0113	-.0662	-.0662	.0671
30.0	-.0505	-.0505	.0106	.0106	.0516
40.0	.0166	.0166	.0376	.0376	.0411
50.0	.0301	.0302	-.0223	-.0223	.0375

$\alpha = 1.5$ / t	$\mathrm{Re}\left(\frac{U_r}{d}\right)^1$	$\mathrm{Re}\left(\frac{U_r}{d}\right)^2$	$\mathrm{Im}\left(\frac{U_r}{d}\right)^1$	$\mathrm{Im}\left(\frac{U_r}{d}\right)^2$	$\left\lvert\frac{U_r}{d}\right\rvert^2$
1.25	-.2845	-.2859	-.1209	-.1214	.3106
2.0	-.0001	-.0003	-.2554	-.2565	.2565
5.0	-.1692	-.1698	.0236	.0236	.1714
10.0	.0385	.0387	-.0733	-.0737	.0833
20.0	-.0660	-.0662	.0370	.0373	.0760
30.0	.0654	.0656	-.0026	-.0027	.0656
40.0	-.0467	-.0469	-.0249	-.0250	.0532
50.0	.0190	.0191	.0398	.0400	.0443

Table 3. First and Second Variational Approximations for U_r. ($ak = 2.0, 2.5, \gamma^2 = 1/3$)

ak = 2.0 / t	$\mathrm{Re}\left(\frac{U_r}{d}\right)^1$	$\mathrm{Re}\left(\frac{U_r}{d}\right)^2$	$\mathrm{Im}\left(\frac{U_r}{d}\right)^1$	$\mathrm{Im}\left(\frac{U_r}{d}\right)^2$	$\left\lvert\frac{U_r}{d}\right\rvert^2$
1.25	-.2081	-.2132	-.2535	-.2642	.3395
2.0	.2172	.2251	-.1897	-.1942	.2973
5.0	.0793	.0831	-.1240	-.1249	.1500
10.0	-.0822	-.0837	-.0967	-.0998	.1302
20.0	.0542	.0555	.0405	.0428	.0701
30.0	-.0625	-.0639	-.0159	-.0172	.0662
40.0	.0529	.0545	.0101	.0109	.0556
50.0	-.0452	-.0467	.0087	.0082	.0474

ak = 2.5 / t	$\mathrm{Re}\left(\frac{U_r}{d}\right)^1$	$\mathrm{Re}\left(\frac{U_r}{d}\right)^2$	$\mathrm{Im}\left(\frac{U_r}{d}\right)^1$	$\mathrm{Im}\left(\frac{U_r}{d}\right)^2$	$\left\lvert\frac{U_r}{d}\right\rvert^2$
1.25	.0384	.0946	-.2507	-.2670	.2832
2.0	.2303	.2296	.0899	.1344	.2660
5.0	.0052	-.0114	.0392	.0275	.0297
10.0	-.0493	-.0494	.0151	-.0002	.0494
20.0	.0105	.0209	-.0435	-.0392	.0445
30.0	.0247	.0174	.0276	.0336	.0378
40.0	-.0327	-.0345	.0072	-.0008	.0345
50.0	.0092	.0163	-.0282	-.0262	.0308

Table 4. First and Second Variational Approximations for U_r.
($ak = 3.0,\ 4.0,\ \gamma^2 = 1/3$)

ak = 3.0 / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_r}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_r}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_r}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_r}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_r}{d}\right\rvert$
1.25	.0427	.0307	.0709	.1398	.1431
2.0	.0020	-.0330	.0388	.0000	.0330
5.0	.0120	.0109	.0908	.1190	.1195
10.0	-.0020	.0085	-.0331	-.0533	.0540
20.0	.0023	.0159	.0144	.0139	.0211
30.0	-.0024	.0005	.0189	.0301	.0301
40.0	-.0074	-.0152	.0010	.0075	.0169
50.0	-.0039	-.0113	-.0120	-.0166	.0201

ak = 4.0 / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_r}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_r}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_r}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_r}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_r}{d}\right\rvert$
1.25	-.3341	-.3555	-.1126	-.1208	.3755
2.0	.2140	.2295	.1540	.1562	.2776
5.0	.0718	.0778	.1362	.1446	.1642
10.0	-.1054	-.1090	-.0990	-.1032	.1501
20.0	-.0890	-.0932	-.0245	-.0260	.0968
30.0	-.0719	-.0754	.0120	.0129	.0765
40.0	-.0534	-.0556	.0380	.0400	.0684
50.0	-.0267	-.0277	.0538	.0562	.0627

Table 5. First and Second Variational Approximations for U_r. ($ak = 5.0$, $\gamma^2 = 1/3$)

$ak = 5.0$ / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_r}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_r}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_z}{d}\right\rvert$
1.25	-.0756	-.0810	-.3919	-.3986	.4068
2.0	-.1982	-.1958	.2034	.2057	.2840
5.0	.1988	.1996	-.0074	-.0076	.1998
10.0	-.0531	-.0530	.1298	.1308	.1412
20.0	.0959	.0968	-.0196	-.0199	.0988
30.0	-.0568	-.0575	-.0556	-.0561	.0803
40.0	-.0122	-.0122	.0675	.0682	.0693
50.0	.0559	.0565	-.0248	-.0251	.0618

Table 6. First and Second Variational Approximations for U_z. ($ak = .2, .5, \gamma^2 = 1/3$)

ak = .2 / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_z}{d}\right\rvert$
1.25	.5829	.5828	.0651	.0650	.5864
2.0	.3169	.3169	.1036	.1036	.3334
5.0	.0623	.0623	.1144	.1144	.1303
10.0	-.0454	-.0454	.0510	.0510	.0683
20.0	-.0020	-.0020	-.0395	-.0395	.0396
30.0	.0238	.0238	.0188	.0188	.0304
40.0	-.0251	-.0251	.0056	.0056	.0257
50.0	.0100	.0100	-.0208	-.0208	.0230

ak = .5 / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_z}{d}\right\rvert$
1.25	.5611	.5599	.1611	.1610	.5826
2.0	.2347	.2347	.2380	.2379	.3341
5.0	-.1241	-.1241	.0660	.0660	.1406
10.0	.0643	.0643	-.0550	-.0550	.0846
20.0	.0121	.0121	-.0560	-.0560	.0573
30.0	-.0271	-.0271	-.0423	-.0423	.0502
40.0	-.0429	-.0429	-.0064	-.0064	.0434
50.0	-.0315	-.0315	.0237	.0237	.0394

Table 7. First and Second Variational Approximations for U_z.
($ak = 1.0,\ 1.5,\ \gamma^2 = 1/3$)

$ak = 1.0$ / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_z}{d}\right\rvert$
1.25	.4810	.4772	.3093	.3081	.5680
2.0	-.0090	-.0088	.3362	.3359	.3360
5.0	.0760	.0759	-.1468	-.1468	.1652
10.0	-.0179	-.0179	-.1112	-.1112	.1126
20.0	-.0836	-.0836	.0192	.0193	.0858
30.0	.0233	.0233	.0708	.0708	.0746
40.0	.0590	.0590	-.0273	-.0273	.0650
50.0	-.0254	-.0254	-.0534	-.0534	.0592

$ak = 1.5$ / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_z}{d}\right\rvert$
1.25	.3382	.3327	.4208	.4194	.5353
2.0	-.2532	-.2520	.2147	.2159	.3318
5.0	.0283	.0284	.1831	.1829	.1851
10.0	-.1381	-.1381	-.0442	-.0441	.1450
20.0	.0653	.0654	.0871	.0871	.1089
30.0	-.0068	-.0068	-.0886	-.0887	.0889
40.0	-.0381	-.0380	.0645	.0645	.0749
50.0	.0608	.0608	-.0267	-.0267	.0664

Table 8. First and Second Variational Approximations for U_z. ($ak = 2.0,\ 2.5,\ \gamma^2 = 1/3$)

$ak = 2.0$ / t	$Re\left(\frac{\overset{1}{U}_z}{d}\right)$	$Re\left(\frac{\overset{2}{U}_z}{d}\right)$	$Im\left(\frac{\overset{1}{U}_z}{d}\right)$	$Im\left(\frac{\overset{2}{U}_z}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_z}{d}\right\rvert$
1.25	.1362	.1230	.4245	.4340	.4511
2.0	-.2869	-.2891	-.0673	-.0605	.2953
5.0	-.1178	-.1180	-.1442	-.1437	.1859
10.0	-.1056	-.1055	.1038	.1038	.1480
20.0	.0652	.0651	-.0938	-.0938	.1142
30.0	-.0277	-.0276	.0847	.0847	.0891
40.0	.0068	.0068	-.0787	-.0787	.0790
50.0	.0083	.0083	.0686	.0686	.0691

$ak = 2.5$ / t	$Re\left(\frac{\overset{1}{U}_z}{d}\right)$	$Re\left(\frac{\overset{2}{U}_z}{d}\right)$	$Im\left(\frac{\overset{1}{U}_z}{d}\right)$	$Im\left(\frac{\overset{2}{U}_z}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_z}{d}\right\rvert$
1.25	.0718	.0071	.2245	.2103	.2104
2.0	-.0368	-.0506	-.1354	-.1372	.1462
5.0	.1014	.1013	.0808	.0801	.1292
10.0	-.0144	-.0143	.1000	.1000	.1010
20.0	-.0438	-.0440	-.1068	-.1069	.1155
30.0	.0491	.0491	-.0324	-.0324	.0589
40.0	.0035	.0035	.0517	.0517	.0518
50.0	-.0436	-.0436	-.0186	-.0186	.0474

Table 9. First and Second Variational Approximations for U_z. (ak = 3.0, 4.0, $\gamma^2 = 1/3$)

ak = 3.0 / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_z}{d}\right\rvert$
1.25	.2729	.2765	.2814	.2065	.3451
2.0	-.1559	-.1598	.0048	.0006	.1598
5.0	.0391	.0393	.0025	.0021	.0393
10.0	-.0135	-.0135	-.0211	-.0211	.0251
20.0	.0166	.0166	-.0212	-.0212	.0270
30.0	.0175	.0175	.0023	.0023	.0177
40.0	-.0004	-.0004	.0161	.0161	.0161
50.0	-.0137	-.0137	.0070	.0070	.0154

ak = 4.0 / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_z}{d}\right\rvert$
1.25	-.0168	-.0141	.5592	.5656	.5658
2.0	.1506	.1520	-.3621	-.3742	.4039
5.0	.2274	.2279	-.1280	-.1289	.2618
10.0	-.1368	-.1369	.1367	.1370	.1936
20.0	-.0417	-.0417	.1268	.1270	.1336
30.0	.0172	.0173	.1097	.1097	.1110
40.0	.0578	.0578	.0771	.0771	.0964
50.0	.0781	.0781	.0371	.0371	.0865

Table 10. First and Second Variational Approximations for U_z. ($ak = 5.0$, $\gamma^2 = 1/3$)

$ak = 5.0$ / t	$\mathrm{Re}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Re}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{1}{U}_z}{d}\right)$	$\mathrm{Im}\left(\frac{\overset{2}{U}_z}{d}\right)$	$\left\lvert\frac{\overset{2}{U}_z}{d}\right\rvert$
1.25	-.3020	-.3307	.2294	.2389	.4080
2.0	.2440	.2482	.2606	.2582	.3581
5.0	-.0192	-.0198	-.2712	-.2715	.2722
10.0	.2852	.2844	-.4133	-.4142	.5024
20.0	-.0295	-.0297	-.1389	-.1389	.1420
30.0	-.0797	-.0796	.0813	.0814	.1139
40.0	.0983	.0983	.0162	.0162	.0997
50.0	-.0358	-.0359	-.0815	-.0815	.0890

APPENDIX F

FIGURES

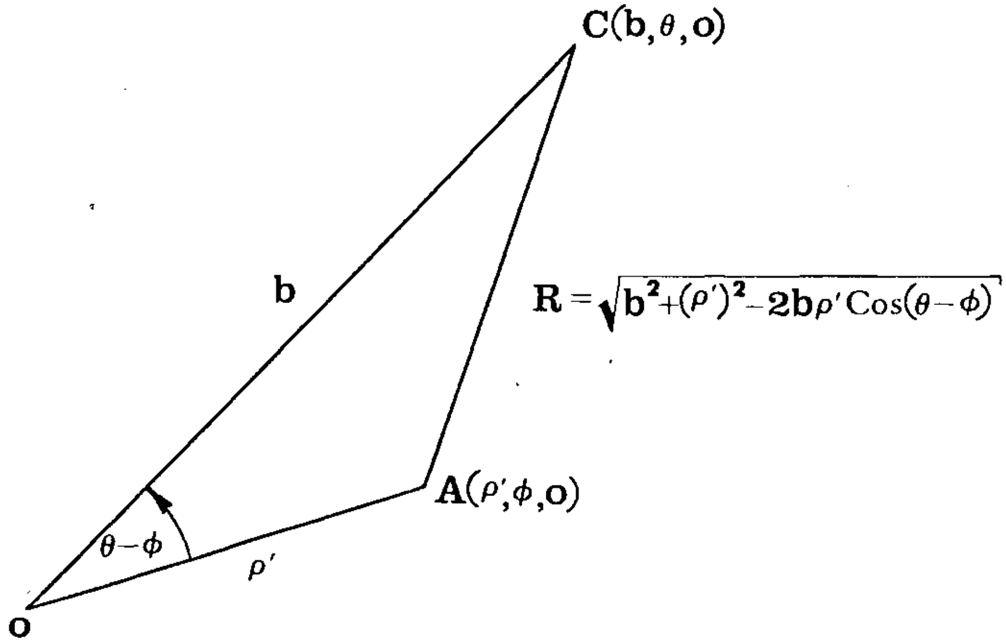

Figure 1. Geometry Used in the Derivation of Equation (23).

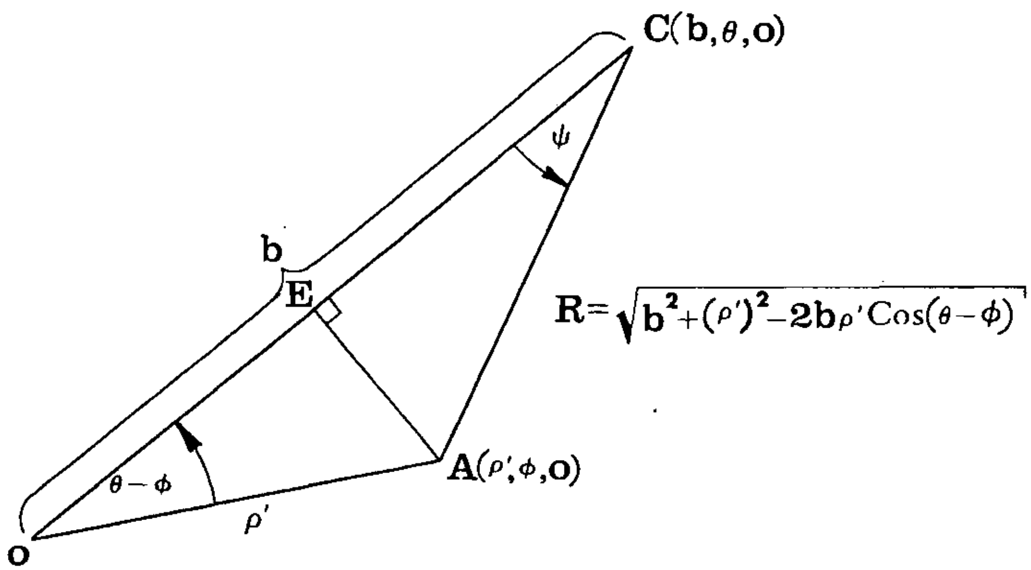

Figure 2. Geometry Used in the Derivation of Equation (37).

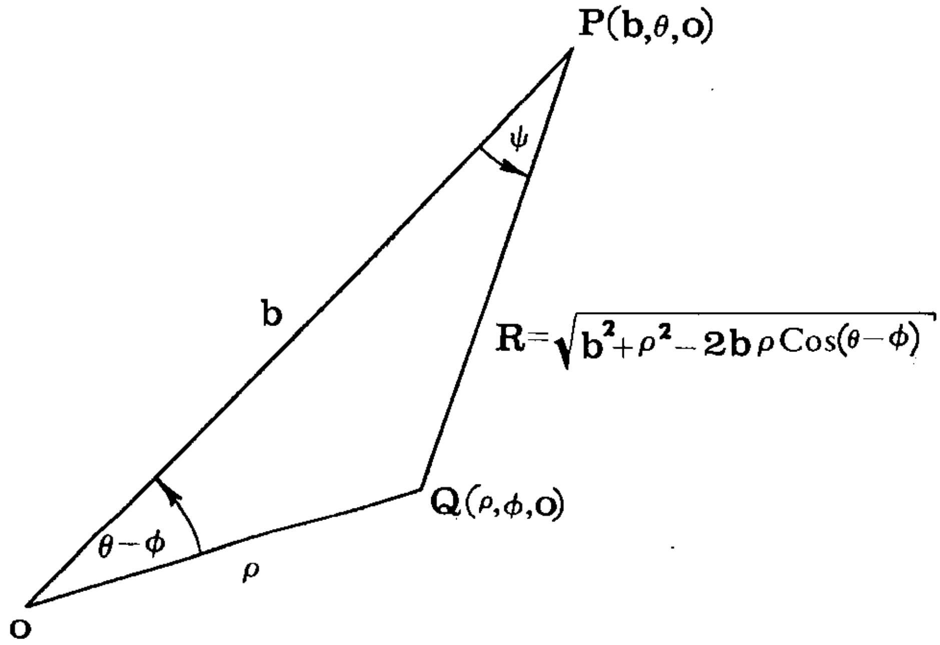

Figure 3. Geometry Used in the Derivation of Equations (155) and (162).

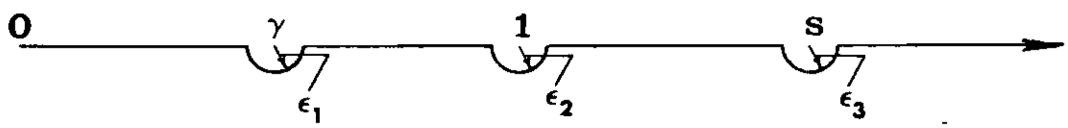

Figure 4. The Radiation Contour.

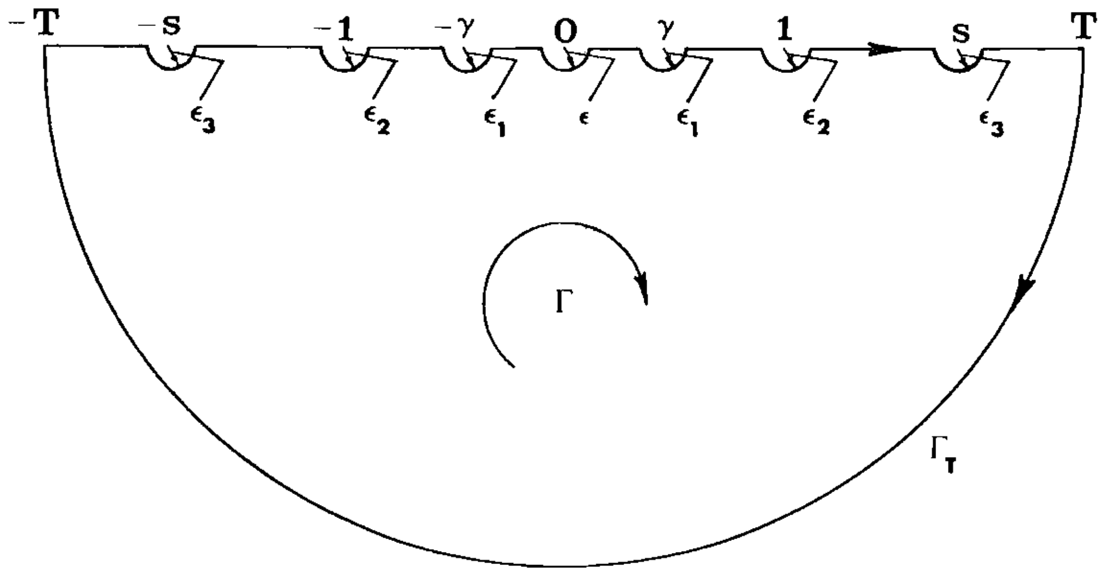

Figure 5. The Contour Γ.

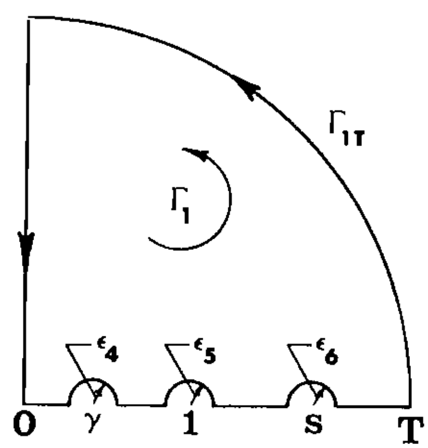

Figure 6. The Contour Γ_1.

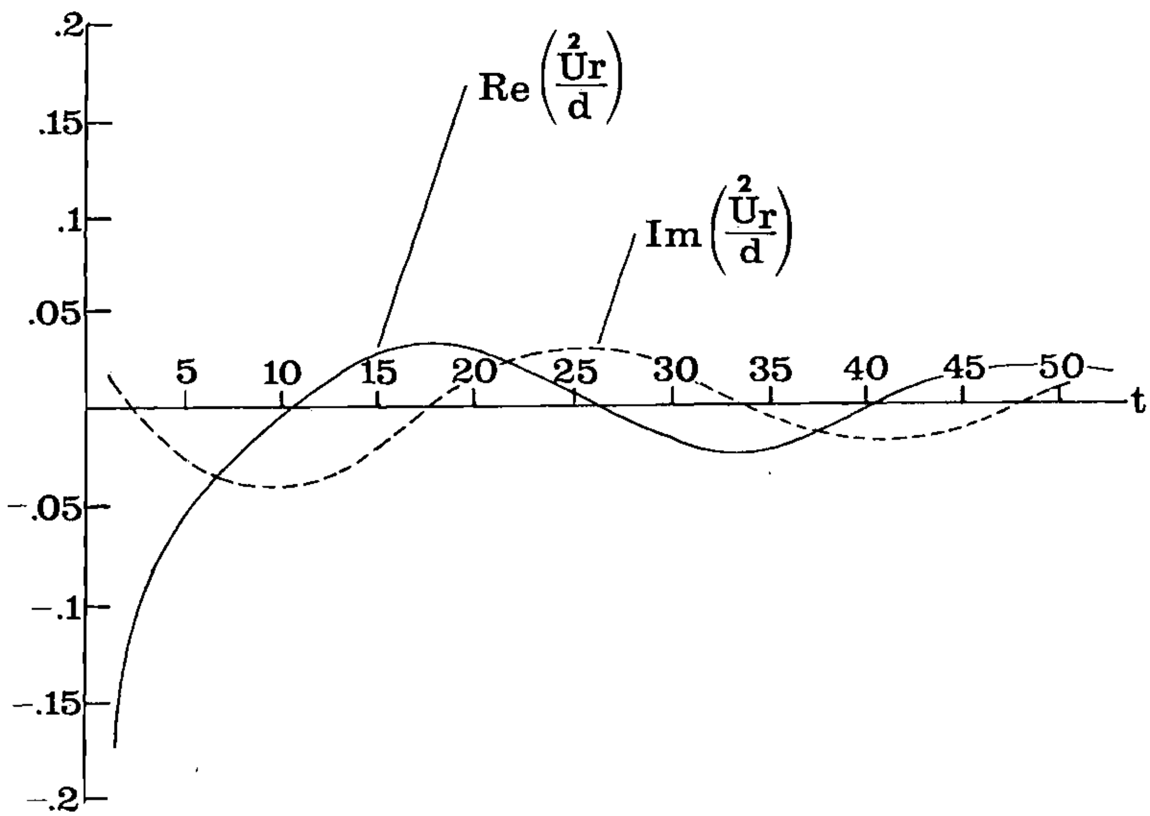

Figure 7. Graphs of Re$\left(\frac{\overset{2}{U}r}{d}\right)$ and Im$\left(\frac{\overset{2}{U}r}{d}\right)$, ak = .2, $\gamma^2 = \frac{1}{3}$.

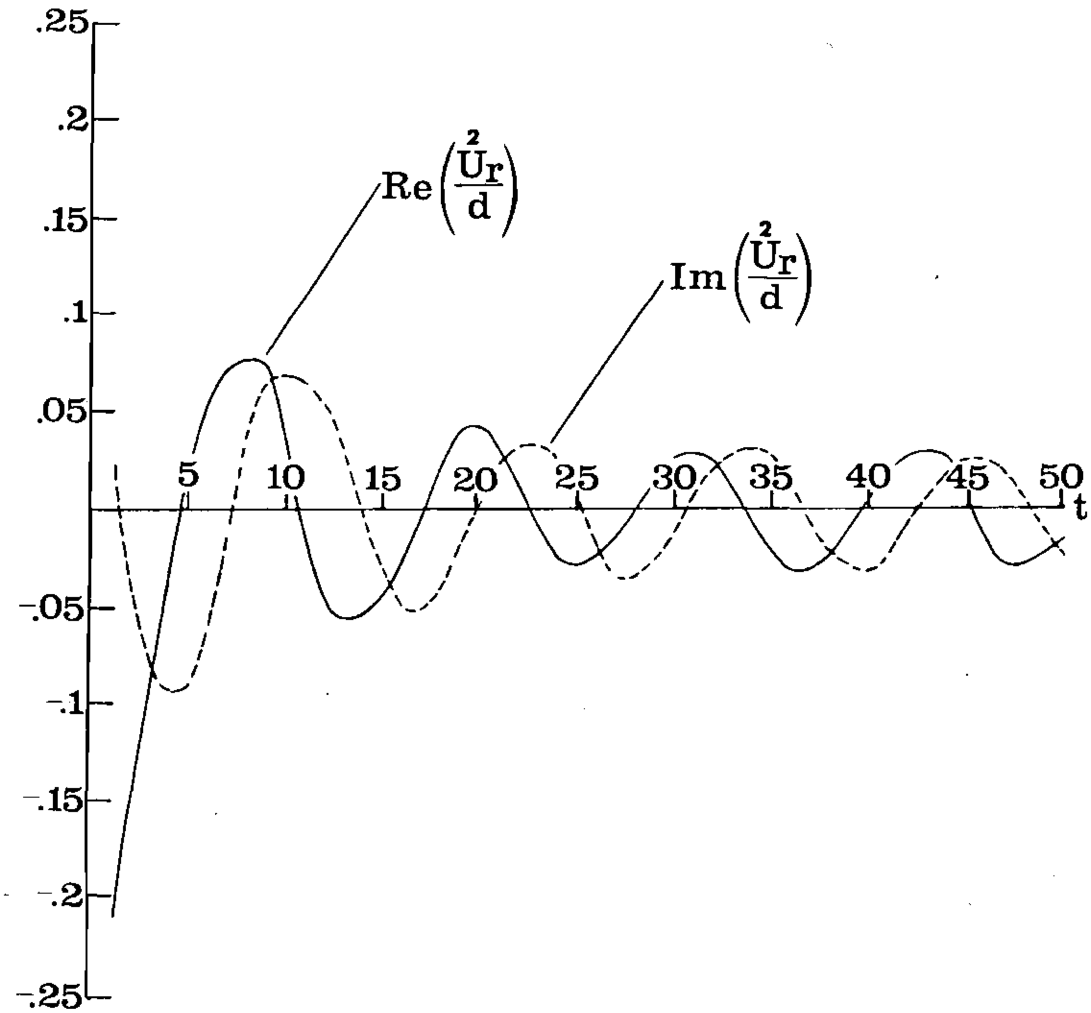

Figure 8. Graphs of $\mathrm{Re}\left(\frac{\overset{2}{U}_r}{d}\right)$ and $\mathrm{Im}\left(\frac{\overset{2}{U}_r}{d}\right)$, $ak = .5$, $\gamma^2 = \frac{1}{3}$.

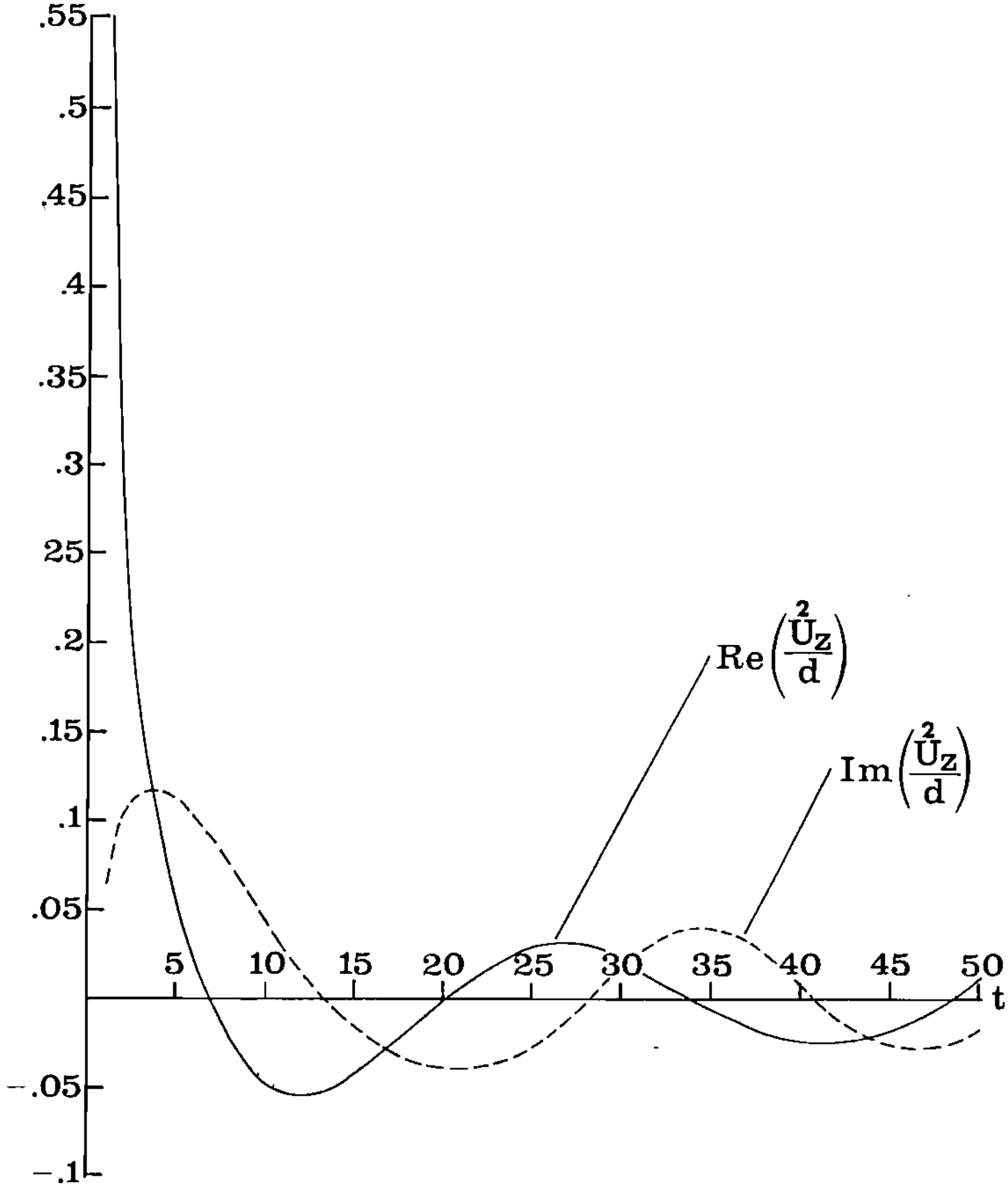

Figure 9. Graphs of $Re\left(\frac{\overset{2}{U}_z}{d}\right)$ and $Im\left(\frac{\overset{2}{U}_z}{d}\right)$, $ak = .2$, $\gamma^2 = \frac{1}{3}$.

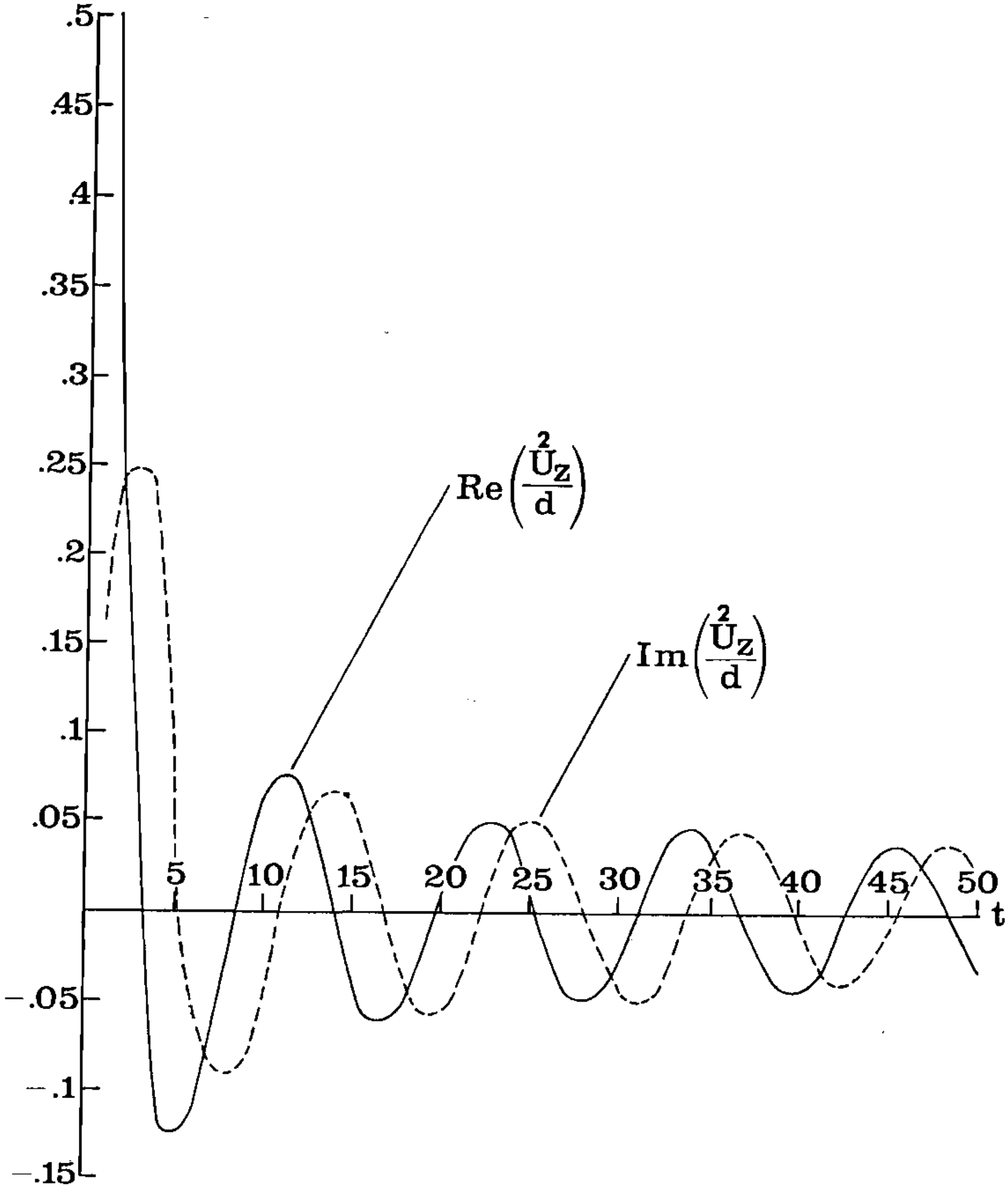

Figure 10. Graphs of $\mathrm{Re}\left(\frac{\overset{2}{U}_z}{d}\right)$ and $\mathrm{Im}\left(\frac{\overset{2}{U}_z}{d}\right)$, $ak = .5$, $\gamma^2 = \frac{1}{3}$.

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